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ELECTROMAGNETIC ENERGY-MOMENTUM TENSOR WITHIN MATERIAL MEDIA

1. MINKOWSKI'S TENSOR

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CONTENTS

Synopsis

The electromagnetic energy-momentum tensor inside a material medium is studied, mainly from a phenomenological point of view. The influence from the medium is taken into account by introducing a dielectric constant and a magnetic permeability. In this paper only Minkowski's tensor is studied, since a comparison between the theory and available experiments indicates that this tensor is well suited to describe usual optical phenomena. Other tensor forms will be dealt with in a forthcoming paper. Here deductive formal procedures are employed; in particular, two sets of conditions are given under which Minkowski's tensor is determined uniquely. Further, attention is given to various characteristic effects, such as negative field energy, which are encountered with the use of Minkowski's tensor.

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I. Introduction

1. Presentation of the Problem

The electromagnetic energy-momentum tensor in a material medium represents a problem that has given rise to a very long-lasting discussion. Maxwell's field equations may be written in covariant form as

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0, \quad \partial_\nu H_{\mu\nu} = \frac{1}{c} j_\mu, \tag{1.1}$$

where the antisymmetric field tensors $F_{\mu\nu}$ and $H_{\mu\nu}$ are defined by $(F_{23}, F_{31}, F_{12}) = \boldsymbol{B}$, $(F_{41}, F_{42}, F_{43}) = i\boldsymbol{E}$, $(H_{23}, H_{31}, H_{12}) = \boldsymbol{H}$ and $(H_{41}, H_{42}, H_{43}) = i\boldsymbol{D}$. The four-vector $j_\mu = (\boldsymbol{j}, ic\varrho)$ is the *external* current density, it does not include polarization or magnetization currents.

By means of the field equations the energy-momentum tensor can easily be constructed if one knows the four-force density in some inertial system. This is the case for an electromagnetic field in vacuum interacting with incoherent matter, the four-current density of which be given by j_μ. In that case the four-force density is given by $f_\mu = (1/c) F_{\mu\nu} j_\nu$ in any reference frame K, since in K^0 – the frame in which the matter under consideration is at rest – the force takes the form $f_\mu^0 = (\varrho^0 \boldsymbol{E}^0, 0)$. Thus $f_\mu = -\partial_\nu S_{\mu\nu}$, where the energy-momentum tensor $S_{\mu\nu}$ is determined by means of (1.1) as

$$S_{\mu\nu} = F_{\mu\alpha} F_{\nu\alpha} - \tfrac{1}{4} \delta_{\mu\nu} F_{\alpha\beta} F_{\alpha\beta} \tag{1.2}$$

since, in this case, $F_{\mu\nu} = H_{\mu\nu}$.

In ponderable bodies, however, it is well known that the force expression is not so easily constructed. If we use (1.1) to form the expression

$$\frac{1}{c} F_{\mu\nu} j_\nu + \tfrac{1}{4}(F_{\alpha\nu} \partial_\mu H_{\alpha\nu} - H_{\alpha\nu} \partial_\mu F_{\alpha\nu}) = -\partial_\nu S^M_{\mu\nu} \tag{1.3}$$

where

$$S^M_{\mu\nu} = F_{\mu\alpha} H_{\nu\alpha} - \tfrac{1}{4} \delta_{\mu\nu} F_{\alpha\beta} H_{\alpha\beta}, \tag{1.4}$$

we see that $S_{\mu\nu}^M$ may be interpreted as an energy-momentum tensor. This is the proposal put forward by H. MINKOWSKI. According to his view, the left hand side of (1.3) is interpreted as the force density within matter.

The expression (1.4) leads to the following interpretation

$$-S_{ik}^M = E_i D_k + H_i B_k - \tfrac{1}{2}\delta_{ik}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}) \tag{1.5a}$$

$$S_{4k}^M = \frac{i}{c} S_k^M = i(\boldsymbol{E}\times\boldsymbol{H})_k, \quad S_{k4}^M = icg_k^M = i(\boldsymbol{D}\times\boldsymbol{B})_k \tag{1.5b}$$

$$-S_{44}^M = W^M = \tfrac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}) \quad (i,k = 1,2,3), \tag{1.5c}$$

where $\boldsymbol{S}^M, \boldsymbol{g}^M, W^M$ denote the energy flux, momentum density and energy density in any frame K.

The connection between the components of the field tensors $F_{\mu\nu}^0$ and $H_{\mu\nu}^0$ in the rest frame K^0 can, in the absence of dispersion, be written as $D_i^0 = \varepsilon_{ik} E_k^0$, $B_i^0 = \mu_{ik} H_k^0$, where ε_{ik} and μ_{ik} are the tensors of dielectric and magnetic permeability. (Dispersion effects are always present, but they are not of essential importance for the present problem and shall therefore simply be omitted.) Now the most important application of the phenomenological theory is in connection with optical phenomena, where ε_{ik} and μ_{ik} are real quantities. In the following we shall always assume ε_{ik} and μ_{ik} to be real. Further, we shall consider only *homogeneous* bodies, such that the gradients of ε_{ik} or μ_{ik} will differ from zero only in the boundary layers. It can readily be verified that in the interior domain of a homogeneous body the second term to the left in (1.3) vanishes, such that

$$\partial_\nu S_{\mu\nu}^M = 0 \tag{1.5d}$$

for optical phenomena ($j_\mu = 0$).

Then define the angular momentum by the quantities

$$M_{\mu\nu} = \int (x_\mu g_\nu - x_\nu g_\mu) dV, \tag{1.6}$$

where $g_\mu = -(i/c) S_{\mu 4}$. When the electromagnetic system is limited in space, it follows from (1.6) that

$$d/dt M_{\mu\nu} = \int (x_\nu f_\mu - x_\mu f_\nu + S_{\nu\mu} - S_{\mu\nu})\, dV, \tag{1.7}$$

where $f_\mu = -\partial_\nu S_{\mu\nu}$. Now consider a finite radiation field enclosed within a homogeneous body at rest, and insert Minkowski's tensor $S_{\mu\nu}^M$ into (1.7).

If the body is optically anisotropic, we obtain even in the frame K^0 an expression for $d/dt^0 M_{ik}^{M0}$ which is different from zero. If the body is optically isotropic, we find $d/dt^0 M_{ik}^{M0} = 0$ since S_{ik}^{M0} is symmetrical when $\boldsymbol{D}^0 = \varepsilon \boldsymbol{E}^0$, $\boldsymbol{B}^0 = \mu \boldsymbol{H}^0$. In another system of reference, however, we have in general $S_{ik}^M \neq S_{ki}^M$ also for isotropic bodies, and thus $d/dt M_{ik}^M \neq 0$. As a conclusion, we find both for anisotropic and isotropic media that an asymmetric mechanical energy-momentum tensor is necessary to achieve balance of the total (field and mechanical) angular momentum. This circumstance has sometimes been felt to be a real difficulty for Minkowski's theory.

Besides, Minkowski's tensor seems to get into conflict with Planck's principle of inertia of energy, as expressed by the relation $\boldsymbol{S} = c^2 \boldsymbol{g}$.

To overcome the difficulties just mentioned, various other proposals of an electromagnetic energy-momentum tensor have been put forward, the best known of which is due to M. Abraham.

For a general introduction to the subject—and for references to some original papers—we refer to Møller's book.(1)

2. Summary and Survey of the Subsequent Work

To facilitate the reading of some of the detailed expositions in the following, we shall in this section give a survey of what follows, and mention some results.

In this paper, which will be followed by a second one on the subject, we shall limit ourselves to a study of Minkowski's tensor. From the phenomenological point of view we are adopting, this tensor is found to be adequate for the description of the usual electromagnetic phenomena, as for instance the situation where an optical wave travels through transparent matter at rest. Comparison with experiments plays an important role in the investigation. But we stress already now that the experimental results do not exclude other possible forms of the electromagnetic energy-momentum tensor; the essential point is rather that Minkowski's form adapts itself to the experiments in a very *simple* way.

The long-lasting discussion on the subject has given rise to an extensive literature, and it appears that in previous phenomenological treatments mainly two lines of attack have been followed. In the first place one uses a deductive method and constructs the energy-momentum tensor on the basis of commonly accepted quantities, for instance the energy in electrostatic and magnetostatic fields, or the (macroscopic) field Lagrangian. In the second place one examines the consequences of using the various tensor forms in appropriate physical situations, and compares with results that can be expected on physical grounds. In these two papers we shall deal with topics connected with both methods of approach.

Let us now review the individual sections. Chapter II is devoted to deductive, and mainly formal, procedures. We start in section 3 by considering a variational method which is applicable to the case of static fields, and which in general leads to the force density and stress tensor when the energy density is known. For the latter density in the electrostatic case, we use the common expression $\frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D}$. Minkowski's tensor is different from other tensor forms proposed even in the case of an electrostatic field in an anisotropic medium, and some contradictory results have appeared in the literature by the use of this method. We show how Minkowski's tensor is one of the legitimate alternatives that result from the formalism, and illustrate the considerations by an example that involves detectable torques on an anisotropic dielectric sphere. An important point is that we shall have the opportunity to make an explicit statement of a crucial assumption which must be imposed if the formalism shall yield Minkowski's tensor. This is the assumption that each volume element experiences a torque density equal to $\boldsymbol{D}\times\boldsymbol{E}$, even if the force on the element is zero.

In section 4 we use this assumption (the "dipole model") as one of the initial conditions in a formal uniqueness proof of the energy-momentum tensor. The dipole model corresponds to a certain requirement on the non-diagonal components of the energy-momentum tensor, and to a vanishing ordinary force density in charge-free homogeneous regions. We require that all components of the four-force density shall vanish, and that the tensor shall be a bilinear form in the field quantities. With these initial conditions, we are led to Minkowski's tensor as the unique result.

Section 5 is devoted to a formal procedure along similar lines as in section 4, but with different initial conditions. In this case relativistic considerations are also involved. We require the energy-momentum tensor to be a bilinear form which is divergence-free and an explicit function of the field quantities $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{B}$ in any inertial frame (but not an explicit function of the four-velocity of the medium). Both anisotropic and isotropic homogeneous media are included in the description. We find that the above-mentioned conditions, in addition to the fact that ε_{ik} and μ_{ik} are symmetric quantities, determine Minkowski's tensor uniquely. In the procedure we use ideas from the corresponding proof for the vacuum-field case, presented by V. Fock.

In order to understand the underlying physical mechanism of wave propagation, it seems desirable as well to examine simple physical situations. In chapter III we undertake this task and construct the electromagnetic energy-momentum tensor in K^0 from semi-phenomenological arguments in the following way: The stress tensor and energy density are taken to be the sum of the two parts corresponding to the electrostatic and magnetostatic cases. Further, we use the fact that the fourth component of the four-force vanishes when an electromagnetic wave travels through a non-absorptive medium. From the continuity equation for energy we can then find the Poynting vector $\boldsymbol{S} = c(\boldsymbol{E} \times \boldsymbol{H})$ and hence the electromagnetic momentum density $\boldsymbol{g} = (1/c)(\boldsymbol{E} \times \boldsymbol{H})$ from Planck's principle of inertia of energy, which is assumed to be valid also for the electromagnetic field in matter. The stress and momentum components determined so far lead to a force density whose effect *may* be to excite a small mechanical momentum of the constituent particles (dipoles). By comparing with a radiation pressure experiment due to R. V. Jones and J. C. S. Richards we find that this suggestion is in fact supported. Corresponding to the mechanical momentum there is a small transport of mechanical energy which, however, together with the rest energy itself, is included in the mechanical part of the total energy-momentum tensor. The conclusion is that Minkowski's tensor gives an adequate description of the propagating wave.

In section 7, some attention is given to the microscopical method of approach. Some difficulties for the acceptance of Minkowski's tensor, which have arisen from microscopical considerations, are discussed. It is stressed that the ambiguity inherent in the formalism is not removed upon transition to the microscopical theory.

In chapter IV we consider methods and specific effects connected with relativity, and limit ourselves to the case of isotropic media. We start in section 8 with a Lagrangian method which involves the use of Noether's theorem, such that the canonical energy-momentum tensor is obtained by a symmetry transformation. Minkowski's tensor is closely connected with the canonical tensor, although the canonical procedure does not rule out other tensor forms. In section 9 we analyse the well-known criterion due to von Laue and Møller on the transformation property of the velocity of the energy in a light wave. By comparing with the Fizeau experiment involving the velocity of light in moving media it is argued that the transformation criterion ought to be fulfilled for an electromagnetic energy-momentum tensor which shall describe the whole light wave. It is a satisfactory feature of Minkowski's tensor that it actually fulfils this criterion. A related experiment reported recently, involving the propagation of light through media in an accelerated reference frame, is also considered.

Section 10 deals with a property which has caused difficulties for the acceptance of Minkowski's tensor, namely the appearance of negative electromagnetic energy in certain cases. We find this to be a direct consequence of the state of covariance of the phenomenological theory: One chooses covariant quantities to be compatible with a scheme one has established on physical grounds in some inertial system. Since certain mechanical quantities are counted together with the field quantities, one obtains—when covariance is imposed—a total four-momentum which is space-like. Therefore, by means of (proper) Lorentz transformations, one can find inertial systems where the total field energy is negative. Closely related to these features is the behaviour of the Čerenkov radiation in the inertial system where the radiating particle is at rest: The energy flow vanishes, while the momentum flow is different from zero and corresponds to a force on the particle.

In section 11 we employ an infinitesimal Lorentz transformation as a symmetry transformation in Noether's theorem and show how the formalism readily adjusts itself to angular momentum quantities which are equivalent to those obtained from Minkowski's tensor. The division of the total field angular momentum into coordinate dependent and coordinate independent parts is discussed.

In the last section we introduce the centre of mass of the field in a relativistic manner. It is found that the various centres obtained in different inertial frames do not in general coincide when considered simultaneously in one frame. By considering in the rest frame K^0 the centres of mass obtained by varying the direction and magnitude of the medium velocity $\boldsymbol{v}$, we find that they are located on a circular disk lying perpendicular to the inner angular momentum vector in K^0 with centre at the centre of mass in K^0.

II. A Variational Method. Uniqueness from two Sets of Conditions

3. A Variational Method in the Case of Static Fields

In this chapter we shall follow a rather formal kind of approach. Our main task is to give two different sets of conditions under which Minkowski's tensor is uniquely determined. In the first place, however, we shall in the present section deal with a derivation of the stress tensor and force density when the electrostatic or magnetostatic field energy in K^0 is known. The calculation will be carried through in the electrostatic case. The method is of interest in itself in so far as Minkowski's tensor is different from the other tensor forms that have been proposed even in the electrostatic case for anisotropic media, and the method has been treated to some extent in the literature(2, 3, 4, 5), but the results do not always agree and we shall go into some details. We shall show how Minkowski's tensor is one of the admissible tensors that result from the formalism, and in particular we shall show the underlying assumptions explicitly. This latter result is of interest in relation to the statement of conditions in the next section.

Then consider an electrostatic field in a medium characterized by material constants η_{ik}, where $E_i = \eta_{ik} D_k$.[1] We assume η_{ik} to have remained unchanged at each point during the (infinitely slow) formation of the field. Then we can integrate the work exerted in building up the field, and obtain in the usual way the free energy

$$\mathscr{F} = \tfrac{1}{2}\int \boldsymbol{E}\cdot\boldsymbol{D}\,dV. \tag{3.1}$$

Now let each volume element dV undergo an arbitrary virtual displacement $\boldsymbol{s}$ so slowly that the process can be taken as reversible. Then we can equate the change of free energy to the mechanical work during the displacement. This "energy method" has been somewhat criticised by some authors (see Smith-White's paper(6) with further references), but there should be little doubt that the method is applicable under the above conditions.

From (3.1) we have

$$\delta\mathscr{F} = \int \boldsymbol{E}\cdot\delta\boldsymbol{D}\,dV + \tfrac{1}{2}\int D_i D_k \,\delta\eta_{ik}\,dV. \tag{3.2}$$

The variations of the integrand are taken at fixed points in space. Letting the electric charge density be denoted by ϱ, we obtain by a partial integration

$$\int \boldsymbol{E}\cdot\delta\boldsymbol{D}\,dV = -\int \nabla\Phi\cdot\delta\boldsymbol{D}\,dV = -\int_{\text{cond.}} \Phi\,\delta\boldsymbol{D}\cdot\boldsymbol{n}\,dS + \int \Phi\,\delta\varrho\,dV, \tag{3.3}$$

where the surface integration is taken over the fixed, charged conductors that are supposed to produce the field. On each conductor Φ is a constant, and as the total charge on a conductor does not change under the displacement, the surface term must vanish. Then

[1] In this section we omit the superscript zero on quantities taken in K°.

$$\frac{d\mathscr{F}}{dt} = \int\left(\Phi\frac{\partial\varrho}{\partial t} + \tfrac{1}{2}D_i D_k \frac{\partial\eta_{ik}}{\partial t}\right)dV. \tag{3.4}$$

Applying the continuity equation $\nabla\cdot(\varrho\boldsymbol{u}) + \partial\varrho/\partial t = 0$ ($\boldsymbol{u} = d\boldsymbol{s}/dt$), we have

$$\left.\begin{aligned} \frac{d\mathscr{F}}{dt} &= -\int_{\text{cond.}}\Phi\varrho\boldsymbol{u}\cdot\boldsymbol{n}dS + \int\left(\nabla\Phi\cdot\varrho\boldsymbol{u} + \tfrac{1}{2}D_i D_k\frac{\partial\eta_{ik}}{\partial t}\right)dV = \\ &= \int\left(-\varrho\boldsymbol{E}\cdot\boldsymbol{u} + \tfrac{1}{2}D_i D_k\frac{\partial\eta_{ik}}{\partial t}\right)dV. \end{aligned}\right\} \tag{3.5}$$

It remains to put $\partial\eta_{ik}/\partial t$ in a form which involves the velocity $\boldsymbol{u}$ explicitly. We therefore write

$$\frac{\partial\eta_{ik}}{\partial t} = \frac{d\eta_{ik}}{dt} - \nabla\eta_{ik}\cdot\boldsymbol{u}, \tag{3.6}$$

where the last term corresponds to the fact that, at a given point $\boldsymbol{r}$, there appears matter which was originally at the point $\boldsymbol{r}-\boldsymbol{s}$. The first term to the right in (3.6) corresponds to the change during the displacement of the element, and arises from two effects. Firstly, η_{ik} may change on account of the components of strain in the body. For small deformations one can make a linear expansion

$$d\eta_{ik}/dt = \gamma_{lm}^{ik}\,ds_{lm}/dt, \tag{3.7}$$

where $s_{lm} = \frac{1}{2}(\partial_m s_l + \partial_l s_m)$ is the symmetrical strain tensor. By symmetry arguments the number of the coefficients γ_{lm}^{ik} can be reduced so that only two of them remain in the case where the body originally is isotropic but under small displacements changes its dielectric properties and becomes anisotropic[7]. If the body is a fluid, so that all shearing strains $s_{lm}(l\neq m)$ vanish, then only one of the γ_{lm}^{ik} remains and corresponds to the electrostriction term $\frac{1}{2}\nabla(E^2\varrho_m\partial\varepsilon/\partial\varrho_m)$ (where ϱ_m is the mass density) in the final expression for the force density. However, we shall neglect these strain effects; they have no interest of principle for our problem. One sees also by an integration over the total system that the contribution to the total force from the electrostriction term vanishes.

Secondly, there will be a contribution to $d\eta_{ik}/dt$ because the crystallographic axes corresponding to a volume element dV rotate by an angle $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ relative to the fixed coordinate system. This effect can be evaluated by transforming η_{ik} as a tensor under the infinitesimal rotation $-\varphi$ of the coordinate axes. Thus we find

$$\tfrac{1}{2}\int D_i D_k \frac{d\eta_{ik}}{dt} dV = -\int (\boldsymbol{D} \times \boldsymbol{E}) \frac{d\varphi}{dt} dV. \tag{3.8}$$

So far we have not specified the variations; the angle φ may vary from element to element. But in order to collect the contributions to the free energy variation, we shall need the relation between φ and $\boldsymbol{s}$, and shall from now on assume the variation to consist of a pure rotation of each element about the origin. Hence $\boldsymbol{s} = \varphi \times \boldsymbol{r}$ and $\varphi = \frac{1}{2}\nabla \times \boldsymbol{s}$. When the medium is thus rotated as a rigid body, we see that possible strain effects are not accounted for; however, as mentioned above, these effects are ignored. To make this kind of variation possible, we assume that the fixed, charged conductors are placed in the vacuum outside the dielectric.

Eq. (3.8) now takes the form

$$\left.\begin{aligned} \tfrac{1}{2}\int D_i D_k \frac{d\eta_{ik}}{dt} dV &= -\tfrac{1}{2}\int (\boldsymbol{D} \times \boldsymbol{E})\cdot(\nabla \times \boldsymbol{u})\, dV = \\ = -\tfrac{1}{2} \int_{\text{cond.}} (E_i \boldsymbol{D}\cdot\boldsymbol{n} - D_i \boldsymbol{E}\cdot\boldsymbol{n}) u_i dS &+ \tfrac{1}{2} \int \partial_k (\boldsymbol{E} D_k - E_k \boldsymbol{D})\cdot \boldsymbol{u} dV, \end{aligned}\right\} \tag{3.9}$$

where the surface integral vanishes.

From (3.5), (3.6) and (3.9) we get

$$\frac{d\mathcal{F}}{dt} = \int [-\varrho \boldsymbol{E} - \tfrac{1}{2} D_i D_k \nabla \eta_{ik} + \tfrac{1}{2}\partial_k (\boldsymbol{E} D_k - E_k \boldsymbol{D})]\cdot \boldsymbol{u} dV. \tag{3.10}$$

Equating $-d\mathcal{F}/dt$ to the mechanical work $\int \boldsymbol{f}\cdot\boldsymbol{u} dV$ exerted by the volume forces $\boldsymbol{f}$, we obtain

$$\boldsymbol{f} = \varrho \boldsymbol{E} + \tfrac{1}{2} D_i D_k \nabla \eta_{ik} - \tfrac{1}{2}\partial_k (\boldsymbol{E} D_k - E_k \boldsymbol{D}). \tag{3.11}$$

By Maxwell's equations this means $f_i = -\partial_k S^A_{ik}$, where the tensor S^A_{ik} is defined by[1]

$$S^A_{ik} = -\tfrac{1}{2}(E_i D_k + E_k D_i) + \tfrac{1}{2}\delta_{ik} \boldsymbol{E}\cdot\boldsymbol{D}. \tag{3.12}$$

The interpretation of (3.11) as a force density and (3.12) as a stress tensor is the result found by Lorentz(2), Pockels(3) and Landau and Lifshitz(4). But there exists an effect not yet considered. There may be a torque present in a volume element also when the force on it is zero, and this torque will perform work during the displacement. Denoting the corre-

[1] Actually, S^A_{ik} is equal to Abraham's tensor in the electrostatic case.

sponding torque density by τ, the additional amount to the total work done is $\int \tau \cdot \varphi dV$. This is the case if the difference $\boldsymbol{P} = \boldsymbol{D} - \boldsymbol{E}$ is due to a distribution of electric dipoles in the medium with the density $\boldsymbol{P}$; we may then write $\tau = \boldsymbol{P} \times \boldsymbol{E} = \boldsymbol{D} \times \boldsymbol{E}$, and

$$\int \tau \cdot \frac{d\varphi}{dt} dV = \tfrac{1}{2} \int (\boldsymbol{D} \times \boldsymbol{E}) \cdot (\nabla \times \boldsymbol{u}) dV = \tfrac{1}{2} \int \partial_k (E_k \boldsymbol{D} - \boldsymbol{E} D_k) \cdot \boldsymbol{u} dV. \tag{3.13}$$

Equating $-d\mathscr{F}/dt$ to the total mechanical work done per unit time, we obtain from (3.10) and (3.13)

$$\left.\begin{aligned} &\int [\varrho \boldsymbol{E} + \tfrac{1}{2} D_i D_k \nabla \eta_{ik} + \tfrac{1}{2} \partial_k (E_k \boldsymbol{D} - \boldsymbol{E} D_k)] \cdot \boldsymbol{u} dV = \\ &\qquad = \int [\boldsymbol{f} + \tfrac{1}{2} \partial_k (E_k \boldsymbol{D} - \boldsymbol{E} D_k)] \cdot \boldsymbol{u} dV. \end{aligned}\right\} \tag{3.14}$$

Hence

$$\boldsymbol{f} = \boldsymbol{f}^M = \varrho \boldsymbol{E} + \tfrac{1}{2} D_i D_k \nabla \eta_{ik}, \; f_i^M = -\partial_k S_{ik}^M, \tag{3.15}$$

i. e. Minkowski's force. Of course the deduction leading to (3.15) is not a proof of the correctness of $\boldsymbol{f}^M$. Its validity is based upon the assumption about the distribution of electric dipoles that leads to (3.13), although it should be noted that this assumption seems to be most natural. As a check we can put $\varrho = \nabla \eta_{ik} = 0$ in (3.15), then it follows that $\boldsymbol{f} = 0$, as expected.

Minkowski's force density was obtained by E. Durand in his book[5].

An example

Let us elucidate the preceding considerations by the following example, considered also by Marx and Györgyi[8]. Let a dielectric sphere be located in a homogeneous electrostatic field, for instance between two condenser plates. Assume that the external field is $\boldsymbol{E}^0 = (E_1^0, E_2^0, 0)$, and choose the principal axes of the sphere to coincide with the coordinate axes so that $\varepsilon_{ik} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$. The field in the vacuum outside the sphere is

$$\boldsymbol{E}^{\text{vac}} = \boldsymbol{E}^0 - \frac{1}{4\pi} \nabla \left(\frac{\boldsymbol{p} \cdot \boldsymbol{n}}{r^2} \right), \tag{3.16}$$

the induced field being a dipole field. One has $\boldsymbol{p} = 3V[(\varepsilon_1 - 1)E_1^0/(\varepsilon_1 + 2), (\varepsilon_2 - 1)E_2^0/(\varepsilon_2 + 2), 0]$, where V is the volume of the sphere. Within the sphere $\boldsymbol{E} = [3E_1^0/(\varepsilon_1 + 2), 3E_2^0/(\varepsilon_2 + 2), 0]$.

The components of the body torque $\boldsymbol{N}$ are determined by the angular momentum balance

$$N_l = -d/dt M_{ik} - \sum_c N_l^{(c)}, \tag{3.17}$$

which we imagine to be taken at an instant just after that the external devices, which might be necessary to keep the system fixed, have been removed. In (3.17) $N_l^{(c)}$ is the l'th component of the torque acting on conductor c, and i, k, l is a cyclic combination of indices. By making use of (1.6) and the conservation laws $f_i = -\partial_\nu S_{i\nu}$, we find

$$\left.\begin{aligned} N_l = \int\limits_{\substack{\text{internal}\\\text{body}}} (x_i f_k - x_k f_i + S_{ik} - S_{ki})\, dV + \int\limits_{\substack{\text{body}\\\text{surface}}} [\boldsymbol{r} \times (\boldsymbol{S}_n - \boldsymbol{S}_n^{\text{vac}})]_l\, dS + \\ + \sum_c \int\limits_{(c)} [\boldsymbol{r} \times \boldsymbol{S}_n^{\text{vac}}]_l\, dS - \sum_c N_l^{(c)}. \end{aligned}\right\} \tag{3.18}$$

Here we have introduced $\boldsymbol{S}_n$ as a vector with components $S_{ni} = S_{ik} n_k$, where the normal vector $\boldsymbol{n}$ points outwards from the body and inwards to a conductor. It is apparent that the two last terms in (3.18) compensate each other, so that we are left with an expression for the body torque which agrees with the expression we would obtain by a direct evaluation of the integral in (1.7), with the opposite sign. This should be expected, sine the torque is a *local* effect.

Now return to the dielectric sphere and insert Minkowski's tensor into (3.18). The only non-vanishing component of the torque is

$$\left.\begin{aligned} N_3^M = \int\limits_{\text{body}} (\boldsymbol{D} \times \boldsymbol{E})_3\, dV - a^3 \int\limits_{\text{surface}} (\boldsymbol{n} \times \boldsymbol{E})_3 (\boldsymbol{D} \cdot \boldsymbol{n})\, d\Omega + \\ + a^3 \int\limits_{\text{surface}} (\boldsymbol{n} \times \boldsymbol{E}^{\text{vac}})_3 (\boldsymbol{E}^{\text{vac}} \cdot \boldsymbol{n})\, d\Omega, \end{aligned}\right\} \tag{3.19}$$

where a is the radius of the sphere and $d\Omega$ the element of solid angle. By using spherical coordinates the two last integrals can be evaluated, so that

$$N_3^M = \int\limits_{\text{body}} (\boldsymbol{D} \times \boldsymbol{E})_3\, dV - (\boldsymbol{p} \times \boldsymbol{E}^0)_3 + (\boldsymbol{p} \times \boldsymbol{E}^0)_3 = (\boldsymbol{p} \times \boldsymbol{E}^0)_3. \tag{3.20}$$

(Actually, the compensation of the two last integrals in (3.19) can be verified also by a mere inspection of the boundary conditions.) The result (3.20) could be checked by experiment. As a characteristic feature of Minkowski's tensor, we see that the body surface term in (3.18) vanishes; it is natural to interpret the effect as a volume effect.

As regards the effects considered in this section, the magnetostatic field is analogous to the electrostatic field and requires no special attention.

4. Uniqueness from first Set of Conditions

This section deals with a formal proof. A set of conditions shall be given, from which we shall show that, within a multiplicative factor in the energy density component, Minkowski's tensor must follow uniquely for the electromagnetic (time-dependent) field inside a homogeneous, anisotropic medium at rest[1].

1. Let us first assume that each volume element experiences a torque density $\tau = \boldsymbol{P} \times \boldsymbol{E} = \boldsymbol{D} \times \boldsymbol{E}$ due to the fact that the constituent electric dipoles are not collinear to the field $\boldsymbol{E}$. This we may call the "dipole model", and it was encountered for the first time in connection with eq. (3.13). We may express this requirement in mathematical form by the relation

$$S_{ik} - S_{ki} = E_k D_i - E_i D_k, \tag{4.1}$$

where S_{ik} is the energy-momentum tensor to be determined.

2. Then require the energy-momentum tensor to be divergence-free,

$$\sum_{\beta} \partial_\beta S_{\alpha\beta} = 0, \tag{4.2}$$

the torque being described by the asymmetry only. For simplicity, we put $\mu = 1$. The summation convention is avoided in this section.

3. As the third condition, $S_{\alpha\beta}$ is required to be a bilinear form in the field quantities.

The three quantities $\boldsymbol{E}$, $\boldsymbol{D}$ and $\boldsymbol{H}$ characterize the field, and (4.2) is an algebraic consequence of the field equations and the constitutive relations which read $E_i = \eta_i D_i$ when the coordinate axes are chosen so that the tensor η_{ik} is diagonal. We first suppose that the η_i are all different. It is now convenient to eliminate $\boldsymbol{E}$ and treat $\boldsymbol{D}$ and $\boldsymbol{H}$ as the independent variables, and we can rewrite (4.2) in the form

[1] We mention already now that both sets of conditions automatically exclude Abraham's tensor from consideration.

$$\sum_{i,l}\left(\frac{\partial S_{\alpha i}}{\partial D_l}\frac{\partial D_l}{\partial x_i}+\frac{\partial S_{\alpha i}}{\partial H_l}\frac{\partial H_l}{\partial x_i}\right)+\frac{1}{ic}\sum_{l}\left(\frac{\partial S_{\alpha 4}}{\partial D_l}\frac{\partial D_l}{\partial t}+\frac{\partial S_{\alpha 4}}{\partial H_l}\frac{\partial H_l}{\partial t}\right)=0, \tag{4.3}$$

where the summations run from 1 to 3.

The time derivatives can be eliminated by means of the two Maxwell's equations

$$\left.\begin{aligned}\frac{\partial D_k}{\partial t} &= c\sum_{m,l}\delta_{kml}\frac{\partial H_l}{\partial x_m}\\ \frac{\partial H_k}{\partial t} &= -c\sum_{m,l}\delta_{kml}\frac{\partial E_l}{\partial x_m} = -c\sum_{m,l}\delta_{kml}\eta_l\frac{\partial D_l}{\partial x_m}.\end{aligned}\right\} \tag{4.4}$$

Hence

$$\sum_{i,l}\left(\frac{\partial S_{\alpha i}}{\partial D_l}+i\frac{\partial S_{\alpha 4}}{\partial H_k}\delta_{kil}\eta_l\right)\frac{\partial D_l}{\partial x_i}+\sum_{i,l}\left(\frac{\partial S_{\alpha i}}{\partial H_l}-i\frac{\partial S_{\alpha 4}}{\partial D_k}\delta_{kil}\right)\frac{\partial H_l}{\partial x_i}=0. \tag{4.5}$$

Here k is supposed to take the value that makes δ_{kil} different from zero. Now having used (4.4) and the constitutive relations, we conclude that (4.5) must be algebraic consequences of the remaining Maxwell's equations, hence (4.5) must be of the form

$$A^{\alpha}\sum_i\partial_i D_i+B^{\alpha}\sum_i\partial_i H_i=0. \tag{4.6}$$

Comparing (4.5) with (4.6), we have then ($i = l$)

$$\frac{\partial S_{\alpha 1}}{\partial D_1}=\frac{\partial S_{\alpha 2}}{\partial D_2}=\frac{\partial S_{\alpha 3}}{\partial D_3}. \tag{4.7}$$

Similarly

$$\frac{\partial S_{\alpha 1}}{\partial H_1}=\frac{\partial S_{\alpha 2}}{\partial H_2}=\frac{\partial S_{\alpha 3}}{\partial H_3}. \tag{4.8}$$

When $i \neq l$, it follows that

$$\frac{\partial S_{\alpha i}}{\partial D_l}+i\frac{\partial S_{\alpha 4}}{\partial H_k}\delta_{kil}\eta_l=0 \tag{4.9}$$

and

$$\frac{\partial S_{\alpha i}}{\partial H_l}-i\frac{\partial S_{\alpha 4}}{\partial D_k}\delta_{kil}=0. \tag{4.10}$$

Hence

$$\left.\begin{aligned} \frac{\partial S_{\alpha i}}{\partial D_l}\eta_i + \frac{\partial S_{\alpha l}}{\partial D_i}\eta_l &= 0 \qquad (4.11) \\ \frac{\partial S_{\alpha i}}{\partial H_l} + \frac{\partial S_{\alpha l}}{\partial H_i} &= 0. \qquad (4.12) \end{aligned}\right\} (i \neq l)$$

If in (4.9) we interchange l and k and differentiate with respect to H_l, and compare with (4.10) differentiated with respect to D_k, we get

$$\frac{\partial^2 S_{\alpha 4}}{\partial H_l \partial H_l}\eta_k = \frac{\partial^2 S_{\alpha 4}}{\partial D_k \partial D_k} \quad (l \neq k). \tag{4.13}$$

The discussion hitherto has closely followed the uniqueness proof for $S_{\mu\nu}$ given by V. Fock(9) in the case of an electromagnetic field in vacuum.

Now the assumption of bilinearity of the tensor components, together with the above equations, are sufficient to determine S_{44} within a multiplicative constant. For this component must be a linear combination of E^2, D^2, H^2 and $\boldsymbol{E}\cdot\boldsymbol{D}$ since it is a three-dimensional scalar. Terms involving $\boldsymbol{H}\cdot\boldsymbol{D}$ and $\boldsymbol{H}\cdot\boldsymbol{E}$ are excluded since $\boldsymbol{E}$, $\boldsymbol{D}$ are polar vectors in opposition to $\boldsymbol{H}$, which is an axial vector. These properties are included in the expression

$$S_{44} = \sum_i a_i D_i^2 + bH^2, \tag{4.14}$$

where a_i and b may involve the material constants. From (4.13) one then finds $a_i = b\eta_i$. The constant b is not determined; with our customary choice of units $b = -\frac{1}{2}$, i. e. $S_{44} = -\frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + H^2)$.

Considering now the spatial component S_{ik}, we see that it can contain linear combinations of the terms $E_iE_k, E_iD_k, E_kD_i, D_iD_k$ and H_iH_k. From (4.2) with $\alpha = i$ it follows, since the momentum density is a polar vector, that S_{ik} must be invariant under space inversion. Therefore terms like E_iH_k and D_iH_k cannot be present. Moreover, we can have terms with the unit tensor δ_{ik} multiplied with a scalar, the scalar being of a form like the right hand side of (4.14). We then write

$$\left.\begin{aligned} S_{ik} &= c_1E_iE_k + c_2E_iD_k + c_3D_iE_k + c_4D_iD_k + c_5H_iH_k - \delta_{ik}\left(\sum_{l=1}^{3} d_lD_l^2 + c_6H^2\right) = \\ &= (c_1\eta_i\eta_k + c_2\eta_i + c_3\eta_k + c_4)D_iD_k + c_5H_iH_k - \delta_{ik}\left(\sum_{l=1}^{3} d_lD_l^2 + c_6H^2\right). \end{aligned}\right\} \tag{4.15}$$

The constants c_l and d_l shall not be restricted to be independent of the material; they shall be permitted to contain symmetric terms such as the sum $\eta_1+\eta_2+\eta_3$. From (4.1) we now have $(\eta_i-\eta_k)(c_2-c_3+1) = 0$, which means

$$c_3 = c_2+1. \tag{4.16}$$

From (4.7) with $\alpha = 1$,

$$\left.\begin{aligned} 2(c_1\eta_1^2+c_2\eta_1+c_3\eta_1+c_4-d_1) &= \\ = c_1\eta_1\eta_2+c_2\eta_1+c_3\eta_2+c_4 &= c_1\eta_1\eta_3+c_2\eta_1+c_3\eta_3+c_4. \end{aligned}\right\} \tag{4.17}$$

From the last equation it follows that

$$c_1\eta_1+c_3 = 0. \tag{4.18}$$

With $\alpha = 2$ we get

$$\left.\begin{aligned} 2(c_1\eta_2^2+c_2\eta_2+c_3\eta_2+c_4-d_2) &= \\ = c_1\eta_2\eta_1+c_2\eta_2+c_3\eta_1+c_4 &= c_1\eta_2\eta_3+c_2\eta_2+c_3\eta_3+c_4. \end{aligned}\right\} \tag{4.19}$$

Hence

$$c_1\eta_2+c_3 = 0. \tag{4.20}$$

Comparison of (4.20) with (4.18) gives $c_1 = c_3 = 0$. From (4.16) then $c_2 = -1$. Now (4.17) and (4.19), together with the corresponding equation for $\alpha = 3$, yield

$$c_4 = 2d_1+\eta_1 = 2d_2+\eta_2 = 2d_3+\eta_3. \tag{4.21}$$

If we use (4.11) with $\alpha = 1, i = 1, l = 2$, we obtain

$$\eta_2 c_4 = \eta_1(2d_2+\eta_2), \tag{4.22}$$

which, together with (4.21), is sufficient to determine the constants

$$c_4 = 0, d_1 = -\tfrac{1}{2}\eta_1, d_2 = -\tfrac{1}{2}\eta_2, d_3 = -\tfrac{1}{2}\eta_3. \tag{4.23}$$

We now turn our attention to the terms S_{i4}. As the momentum density is a polar vector, any actual bilinear combination can be written in the form

$$S_{i4} = \sum_{j,\,k}\delta_{ijk}f_jD_jH_k, \tag{4.24}$$

where f_j may contain material constants. Putting $\alpha = 1$ in (4.9), we have

$$\frac{\partial S_{14}}{\partial H_1} = 0,\ \frac{\partial S_{14}}{\partial H_2} = -iD_3,\ \frac{\partial S_{14}}{\partial H_3} = iD_2, \tag{4.25}$$

which are compatible with (4.24), when $f_2 = f_3 = i$. Similar arguments for $\alpha = 2,3$ give $f_1 = i$. Therefore

$$S_{i4} = i\sum_{j,k}\delta_{ijk}D_jH_k = i(\boldsymbol{D}\times\boldsymbol{H})_i. \tag{4.26}$$

Then (4.10) gives, when $\alpha = 1$, $i = 1$, $l = 2$, that $c_6 = -\frac{1}{2}$. With another combination of indices, or from (4.8) or (4.12), one finds $c_5 = -1$. Inserted into (4.15)

$$S_{ik} = -E_iD_k - H_iH_k + \delta_{ik}\tfrac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + H^2). \tag{4.27}$$

In the same way one finds from (4.9) and (4.10) that the remaining components are

$$S_{4i} = i\sum_{j,k}\delta_{ijk}\eta_jD_jH_k = i(\boldsymbol{E}\times\boldsymbol{H})_i. \tag{4.28}$$

We have thus arrived at Minkowski's tensor.

Note that as a result of the linear combination postulated in (4.15), we obtained to a certain extent the dependence on η_i of the coefficient in front of D_iD_k on the right of this equation. If instead we had put the first term of the last expression equal to the general form $c_{ik}D_iD_k$, the equations (4.1) and (4.7—10) would not have been sufficient to determine the components c_{ik} such as given above, where $c_{ik} = c_1\eta_i\eta_k + c_2\eta_i + c_3\eta_k + c_4$.

The foregoing procedure is based on the assumption of different material constants; the conclusions are valid only when $\eta_i - \eta_k \neq 0$. If, however, two of these constants are equal, but different from the third, we see without difficulty that the present treatment need not be changed. In the deduction of (4.16) for instance, we use first the two unequal η_i and η_k to give $c_3 = c_2 + 1$. The same considerations apply when we construct the equations corresponding to (4.18) and (4.20), giving $c_1 = c_3 = 0$, as before. We arrive again at Minkowski's tensor as the final result.

But if the η_i are all equal, our equations are not sufficient to determine the components $S_{i\alpha}$ uniquely. With a simplified expression for S_{ik} corresponding to (4.15) and the assumption (4.24), we can use (4.7—10) and determine the quantities $S_{i\alpha}$, except for a multiplicative constant. This constant comes in addition to the multiplicative factor appearing in the determination of S_{44}. This is connected with the fact that we cannot take advantage of the dipole model in this case; instead, we may take into account that $S_{\alpha\beta}$ is a tensor under Lorentz transformations. These concepts are taken up in the next section.

5. Uniqueness from Second Set of Conditions

In this section we shall give another formal derivation of Minkowski's tensor, based on somewhat different initial conditions.

Let us first refer to the treatment in Fock's book(9), for a consideration of the problem to determine in general an energy-momentum tensor $S_{\alpha\beta}$ uniquely. He takes explicitly into account that $S_{\alpha\beta}$ be a tensor, and he requires it to be symmetric and to have a vanishing four-divergence. However, to determine $S_{\alpha\beta}$ uniquely (or within a constant multiplying factor, provided that suitable conditions exist at infinity), he finds it essential to lean on the requirement that the energy-momentum tensor should be a function of the state of the system. By "state" is meant the following. If the equations of motion and the field equations are written as first order equations for the unknown functions φ_i, the latter functions are said to characterize the state. Any function of φ_i that does not contain their derivatives and also does not contain the coordinates explicitly, is called a function of the state. With this additional conditions imposed, he claims the energy-momentum tensor to be determined in principle for every physical system.

Now our system is different from those considered by Fock since $S_{\alpha\beta}$ must be permitted to be asymmetric. Therefore we shall carry through the proof in detail.

We recall the three initial conditions which were given in the preceding section. Here we shall release the condition 1 and instead require $S_{\alpha\beta}$ to be a function of the electromagnetic state of the system. Since the field equations contain the field quantities only (and not the four-velocity V_μ of the medium explicitly), it follows that $S_{\alpha\beta}$ also contains only field quantities. This is to be true in *any* inertial frame, and we shall use this property explicitly when we perform Lorentz transformations. We have thus

$$S_{\alpha\beta} = S_{\alpha\beta}(\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B}). \tag{5.1}$$

2. The tensor is still required to be divergence-free

$$\sum_\beta \partial_\beta S_{\alpha\beta} = 0, \tag{5.2}$$

where also in this section the summation convention is avoided.

3. The tensor is still required to be a bilinear form.

The material constants are in general ε_{ik} and μ_{ik}. From geometrical consideration of the fact that in K^0 the magnitude of $\boldsymbol{P}^0$ is proportional to that of $\boldsymbol{E}^0$, while the angle between $\boldsymbol{P}^0$ and $\boldsymbol{E}^0$ is constant for a given orientation of the field, it follows that ε_{ik} is symmetric. Similar considerations apply to μ_{ik}. Also isotropic media are now included in the description.

The next task is to show that, within a multiplicative constant, the conditions mentioned are sufficient to yield Minkowski's tensor. It is natural to work directly with the field quantities $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B}$ instead of eliminating some of them by means of the constitutive relations in K^0. Eqs. (5.2) are algebraic consequences of Maxwell's equations

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \times \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} \tag{5.3}$$

$$\nabla \cdot \boldsymbol{D} = 0, \ \nabla \cdot \boldsymbol{B} = 0 \tag{5.4}$$

and the constitutive relations. Eq. (5.1) implies that we have to write the constitutive relations in a form where neither the material constants nor the body velocity is present. The simplest way of eliminating the material constants in K^0 is to write

$$\left.\begin{array}{ll} \boldsymbol{E}^0 \cdot \partial_\mu^0 \boldsymbol{D}^0 - \boldsymbol{D}^0 \cdot \partial_\mu^0 \boldsymbol{E}^0 = 0 & (5.5) \\ \boldsymbol{H}^0 \cdot \partial_\mu^0 \boldsymbol{B}^0 - \boldsymbol{B}^0 \cdot \partial_\mu^0 \boldsymbol{H}^0 = 0, & (5.6) \end{array}\right\} (\mu = 1-4)$$

so that the constitutive equations involve the first order derivatives of the fields, as do Maxwell's equations. Now (5.5) can be written $\sum(F_{4\beta}^0 \partial_\mu^0 H_{4\beta}^0 - H_{4\beta}^0 \partial_\mu^0 F_{4\beta}^0) = 0$, which cannot be brought into a covariant form except by introducing the four-velocity V_μ of the medium. Similarly for (5.6). We therefore try to write the constitutive relations in K as a linear combination of the terms $\sum F_{\alpha\beta} \partial_\mu H_{\alpha\beta}$ and $\sum H_{\alpha\beta} \partial_\mu F_{\alpha\beta}$ and readily find that

$$\sum_{\alpha, \beta} (F_{\alpha\beta} \partial_\mu H_{\alpha\beta} - H_{\alpha\beta} \partial_\mu F_{\alpha\beta}) = 0 \tag{5.7}$$

or

$$\boldsymbol{E} \cdot \partial_\mu \boldsymbol{D} - \boldsymbol{D} \cdot \partial_\mu \boldsymbol{E} + \boldsymbol{H} \cdot \partial_\mu \boldsymbol{B} - \boldsymbol{B} \cdot \partial_\mu \boldsymbol{H} = 0 \tag{5.8}$$

represent the simplest form of the constitutive relations with the required properties.

Let us then write (5.2) in the following form, assuming $S_{\alpha\beta}$ to be a function of the state:

$$\left.\begin{aligned}&\sum_{i,l}\left(\frac{\partial S_{\alpha i}}{\partial E_l}\frac{\partial E_l}{\partial x_i}+\frac{\partial S_{\alpha i}}{\partial D_l}\frac{\partial D_l}{\partial x_i}+\frac{\partial S_{\alpha i}}{\partial H_l}\frac{\partial H_l}{\partial x_i}+\frac{\partial S_{\alpha i}}{\partial B_l}\frac{\partial B_l}{\partial x_i}\right)+\\&+\frac{1}{ic}\sum_{l}\left(\frac{\partial S_{\alpha 4}}{\partial E_l}\frac{\partial E_l}{\partial t}+\frac{\partial S_{\alpha 4}}{\partial D_l}\frac{\partial D_l}{\partial t}+\frac{\partial S_{\alpha 4}}{\partial H_l}\frac{\partial H_l}{\partial t}+\frac{\partial S_{\alpha 4}}{\partial B_l}\frac{\partial B_l}{\partial t}\right)=0,\end{aligned}\right\}\tag{5.9}$$

and demand (5.9) to be algebraic consequences of (5.3), (5.4) and (5.8). By means of (5.3) two of the time derivatives can be eliminated, but the derivatives $\dot{E}_l$ and $\dot{H}_l$ cannot be eliminated by means of Maxwell's equations. The actual equation is then (5.8) with $\mu = 4$, and by comparison with (5.9) we obtain the conditions

$$\frac{\partial S_{\alpha 4}}{\partial E_l} = A^{\alpha} D_l, \quad \frac{\partial S_{\alpha 4}}{\partial H_l} = A^{\alpha} B_l. \tag{5.10}$$

Since $S_{\alpha 4}$ is a bilinear form, the quantity A^{α} must be independent of the fields. Eq. (5.9) now reads

$$\left.\begin{aligned}&\sum_{i,l}\left(\frac{\partial S_{\alpha i}}{\partial E_l}+iA^{\alpha}\delta_{kil}H_k+i\delta_{kil}\frac{\partial S_{\alpha 4}}{\partial B_k}\right)\frac{\partial E_l}{\partial x_i}+\sum_{i,l}\left(\frac{\partial S_{\alpha i}}{\partial H_l}-\right.\\&\left.-iA^{\alpha}\delta_{kil}E_k-i\delta_{kil}\frac{\partial S_{\alpha 4}}{\partial D_k}\right)\frac{\partial H_l}{\partial x_i}+\sum_{i,l}\frac{\partial S_{\alpha i}}{\partial D_l}\frac{\partial D_l}{\partial x_i}+\sum_{i,l}\frac{\partial S_{\alpha i}}{\partial B_l}\frac{\partial B_l}{\partial x_i}=0.\end{aligned}\right\}\tag{5.11}$$

This equation must be a linear combination of the remaining equations (5.4) and (5.8) with $\mu = 1,2,3$. Only linear forms are permissible because we have assumed the condition (5.1), and inspection of (5.11) then shows that only terms linear in the derivatives are present. Hence (5.11) must be of the form

$$\sum_i C^{\alpha i}(\boldsymbol{D}\cdot\partial_i\boldsymbol{E}+\boldsymbol{B}\cdot\partial_i\boldsymbol{H}-\boldsymbol{E}\cdot\partial_i\boldsymbol{D}-\boldsymbol{H}\cdot\partial_i\boldsymbol{B})+F^{\alpha}\nabla\cdot\boldsymbol{D}+G^{\alpha}\nabla\cdot\boldsymbol{B}, \tag{5.12}$$

where the Lagrangian multipliers $C^{\alpha i}, F^{\alpha}$ and G^{α} do not contain differential operators ∂_{μ}. By equating (5.12) to (5.11) we can look upon this new equation as an identity in the derivatives of the fields with respect to the coordinates, because of the presence of the multipliers. Hence we obtain the relations

$$\frac{\partial S_{\alpha i}}{\partial E_l}+iA^{\alpha}\delta_{kil}H_k+i\delta_{kil}\frac{\partial S_{\alpha 4}}{\partial B_k} = C^{\alpha i} D_l \tag{5.13}$$

$$\frac{\partial S_{\alpha i}}{\partial H_l} - iA^\alpha \delta_{kil} E_k - i\delta_{kil} \frac{\partial S_{\alpha 4}}{\partial D_k} = C^{\alpha i} B_l. \tag{5.14}$$

When $i \neq l$,

$$\frac{\partial S_{\alpha i}}{\partial D_l} = -C^{\alpha i} E_l \tag{5.15}$$

$$\frac{\partial S_{\alpha i}}{\partial B_l} = -C^{\alpha i} H_l. \tag{5.16}$$

When $i = l$,

$$\frac{\partial S_{\alpha i}}{\partial D_i} = -C^{\alpha i} E_i + F^\alpha \tag{5.17}$$

$$\frac{\partial S_{\alpha i}}{\partial B_i} = -C^{\alpha i} H_i + G^\alpha. \tag{5.18}$$

Then put $\alpha = 4$, and examine which simplifications can be made in these equations from the requirement of bilinearity of $S_{\alpha\beta}$. If we make a rotation of the spatial coordinate axes in K, we know that $\partial_\beta S_{4\beta}$ remains unchanged, and so the expression (5.12) is also unchanged. Hence, since the expression in the parenthesis in (5.12) transforms as a three-dimensional vector, the quantities C^{4i} must transform similarly. But according to the bilinearity of $S_{\alpha\beta}$, C^{4i} must be independent of the fields, therefore $C^{4i} = 0$. By similar arguments we conclude that $F^4 = G^4 = 0$.

The reduced system of equations we have now obtained is easily solved for the components $S_{4\beta}$. By assuming the form

$$S_{4i} = \sum_{j,k} \delta_{ijk}(a_1 E_j H_k + a_2 E_j B_k + a_3 D_j H_k + a_4 D_j B_k), \tag{5.19}$$

we obtain from (5.15) and (5.16) that $a_2 = a_3 = a_4 = 0$. If we fix the remaining constant $a_1 = i$, we obtain

$$S_{4i} = i(\boldsymbol{E} \times \boldsymbol{H})_i. \tag{5.20}$$

Similarly, by assuming

$$S_{44} = b_1 E^2 + b_2 D^2 + b_3 H^2 + b_4 B^2 + b_5 \boldsymbol{E} \cdot \boldsymbol{D} + b_6 \boldsymbol{H} \cdot \boldsymbol{B}, \tag{5.21}$$

we obtain by virtue of (5.13), (5.14) and (5.10)

$$S_{44} = -\tfrac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}). \tag{5.22}$$

The discussion hitherto is in principle similar to that leading to the components $S_{4\beta}$ in section 4, although the discussion has been carried through for any K. But in order to find the remaining components, we shall use (5.1) and the tensor property of $S_{\alpha\beta}$. We perform an infinitesimal Lorentz transformation $x'_\mu = x_\mu + \sum \omega_{\mu\nu} x_\nu$, where the antisymmetric $\omega_{\mu\nu}$ is given by $\omega_{ik} = \varphi_l(\text{cycl})$, $\omega_{i4} = iu_i/c$. We obtain

$$\left.\begin{aligned} \delta \boldsymbol{E} &= \boldsymbol{E}' - \boldsymbol{E} = \frac{1}{c}(\boldsymbol{u} \times \boldsymbol{B}) - (\varphi \times \boldsymbol{E}) \\ \delta \boldsymbol{D} &= \frac{1}{c}(\boldsymbol{u} \times \boldsymbol{H}) - (\varphi \times \boldsymbol{D}) \\ \delta \boldsymbol{H} &= -\frac{1}{c}(\boldsymbol{u} \times \boldsymbol{D}) - (\varphi \times \boldsymbol{H}) \\ \delta \boldsymbol{B} &= -\frac{1}{c}(\boldsymbol{u} \times \boldsymbol{E}) - (\varphi \times \boldsymbol{B}). \end{aligned}\right\} \tag{5.23}$$

When a system in general is described by a set of functions γ_s, the change of these, on account of the present transformation, can be written as

$$\delta \gamma_s = \tfrac{1}{2} \sum_{\mu,\nu} \omega_{\mu\nu} \Psi_s^{\mu\nu}, \tag{5.24}$$

where the antisymmetric $\Psi_s^{\mu\nu}$ are functions of γ_s. We follow the method given by Fock[(9)] (§ 31*) by introducing a set of operators $X^{\mu\nu}$ by the equations

$$X^{\mu\nu}(h) = \sum_s \Psi_s^{\mu\nu} \frac{\partial h}{\partial \gamma_s}, \tag{5.25}$$

where h is some function of γ_s. Hence

$$X^{\mu\nu}(\gamma_s) = \Psi_s^{\mu\nu}, \tag{5.26}$$

which, inserted into (5.24), gives

$$\delta \gamma_s = \tfrac{1}{2} \sum_{\mu,\nu} \omega_{\mu\nu} X_s^{\mu\nu}(\gamma_s). \tag{5.27}$$

The variation δh can also be expressed in terms of these operators; we have

$$\delta h = \sum_s \frac{\partial h}{\partial \gamma_s} \delta \gamma_s = \tfrac{1}{2} \sum_{s,\mu,\nu} \frac{\partial h}{\partial \gamma_s} \omega_{\mu\nu} \Psi_s^{\mu\nu} = \tfrac{1}{2} \sum_{\mu,\nu} \omega_{\mu\nu} X^{\mu\nu}(h). \tag{5.28}$$

With $h = S_{\alpha\beta}(\gamma_s)$:

$$\delta S_{\alpha\beta} = \tfrac{1}{2}\sum_{\mu,\nu} \omega_{\mu\nu} X^{\mu\nu}(S_{\alpha\beta}). \tag{5.29}$$

This equation is compared with $\delta S_{\alpha\beta}$ obtained from a tensor transformation

$$\delta S_{\alpha\beta} = \sum_{\mu,\nu} \omega_{\mu\nu}(\delta_{\mu\alpha} S_{\nu\beta} + \delta_{\mu\beta} S_{\alpha\nu}), \tag{5.30}$$

and there results

$$X^{\mu\nu}(S_{\alpha\beta}) = \delta_{\mu\alpha} S_{\nu\beta} - \delta_{\nu\alpha} S_{\mu\beta} + \delta_{\mu\beta} S_{\alpha\nu} - \delta_{\nu\beta} S_{\alpha\mu}. \tag{5.31}$$

Finally, from (5.25) and (5.26)

$$X^{\mu\nu}(S_{\alpha\beta}) = \sum_s X^{\mu\nu}(\gamma_s) \frac{\partial S_{\alpha\beta}}{\partial \gamma_s}. \tag{5.32}$$

In our case $\delta\gamma_s$ are given by (5.23), and as S_{44} and S_{4i} are already found, we shall see that the present equations are sufficient to determine the remaining components $S_{i\beta}$. It should be noticed that, as γ_s denote the field quantities, eq. (5.1) is essential for the passage from (5.28) to (5.29). Further, it is essential that $S_{\alpha\beta}$ is a tensor for the establishment of (5.30).

Now compare (5.23) with the general (5.27). There results

$$\left.\begin{aligned} X^{ik}(\gamma_s) &= E_k \frac{\partial}{\partial E_i} - E_i \frac{\partial}{\partial E_k} + D_k \frac{\partial}{\partial D_i} - D_i \frac{\partial}{\partial D_k} + \\ &\quad + H_k \frac{\partial}{\partial H_i} - H_i \frac{\partial}{\partial H_k} + B_k \frac{\partial}{\partial B_i} - B_i \frac{\partial}{\partial B_k} \\ X^{4i}(\gamma_s) &= -i \sum_{j,k} \delta_{ijk} \left(E_i \frac{\partial}{\partial B_k} - B_j \frac{\partial}{\partial E_k} + D_j \frac{\partial}{\partial H_k} - H_j \frac{\partial}{\partial D_k} \right). \end{aligned}\right\} \tag{5.33}$$

From (5.31) we obtain

$$X^{4i}(S_{44}) = S_{i4} + S_{4i}. \tag{5.34}$$

Calculating from (5.32)

$$X^{4i}(S_{44}) = \sum_s \left[X^{4i}(E_s) \frac{\partial S_{44}}{\partial E_s} + X^{4i}(D_s) \frac{\partial S_{44}}{\partial D_s} + X^{4i}(H_s) \frac{\partial S_{44}}{\partial H_s} + X^{4i}(B_s) \frac{\partial S_{44}}{\partial B_s} \right]$$

and using (5.33) and (5.22), we get

$$X^{4i}(S_{44}) = i(\boldsymbol{D} \times \boldsymbol{B} + \boldsymbol{E} \times \boldsymbol{H})_i. \tag{5.35}$$

From (5.35), (5.34) and (5.20) then

$$S_{i4} = i(\boldsymbol{D} \times \boldsymbol{B})_i. \tag{5.36}$$

From (5.31) we have for the spatial components

$$S_{ik} = X^{4i}(S_{4k}) + \delta_{ik} S_{44}. \tag{5.37}$$

From (5.32), (5.33) and (5.20)

$$X^{4i}(S_{4k}) = \sum_s X^{4i}(\gamma_s) \frac{\partial S_{4k}}{\partial \gamma_s} = -E_i D_k - H_i B_k + \delta_{ik}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}), \tag{5.38}$$

and from (5.37) then

$$S_{ik} = -E_i D_k - H_i B_k + \tfrac{1}{2}\delta_{ik}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}). \tag{5.39}$$

The adjustment of the constant a_1 in (5.19) has thus led to Minkowski's expression for all components. It follows that the two sets of assumptions from the preceding section and the present section must be equivalent.

III. Derivation of Minkowski's Tensor by a Semi-Empirical Method

6. Consideration of a Plane Wave Travelling through Matter at Rest

This chapter forms the central part of our work. By using the phenomenological theory and leaning on experiments, we shall construct the electromagnetic energy-momentum tensor in the simple optical situation where a plane light wave travels through a dielectric body at rest. We emphasize that we do not intend to give a *formal* derivation of Minkowski's tensor; we use simple, formal arguments to illustrate what *may* happen, and then take the lacking information from experiments.

Isotropic matter

One might first think of the possibility to use microscopical considerations as a guide to construct an expression for the force density f_μ directly in terms of the macroscopical fields. Some attempts have been made in this direction[10, 11]. We shall study the microscopical line of approach to some extent in the next section, but mention already now that there are some difficulties of principle with a construction of the force density in this way. The macroscopical force can be written as the average over appropriate regions in space and time of the microscopical force acting on the external charges and currents, as well as on the matter itself. But since the force is of the second order in the field quantities, we cannot simply find it in terms of products of the macroscopical fields when the microscopical fields are correlated in an unknown way.

Further, the macroscopic variational method which is applicable in electrostatic and magnetostatic cases commonly fails when the fields are time varying.

Let us then employ the simple macroscopic method followed, for instance, by LANDAU and LIFSHITZ(4). It is usually so that the stress tensor and energy density may be taken as the sum of the parts corresponding to the electrostatic and magnetostatic cases. This is a reasonable construction at frequencies much lower than the eigenfrequencies of the molecular or electronic vibrations which lead to the electric or magnetic polarization of the matter. Then the linear relations between $\boldsymbol{E}, \boldsymbol{D}$ and $\boldsymbol{H}, \boldsymbol{B}$ are still valid, when the fields are not too strong. But the latter relations are valid also in the optical regions where the dielectric permeability is approximately frequency independent in virtue of the electronic polarization, but where the contribution from the slower molecular vibrations is absent. In this optical region we can therefore approximately put the magnetic permeability equal to 1. We assume that the above-mentioned construction of the stress tensor and energy density is valid also in this case, so that these quantities are given by (1.5a) and (1.5c).

As in the former treatment in section 3, we ignore electrostriction and magnetostriction effects.

We then have to determine the remaining components of the energy-momentum tensor $S_{\mu\nu}$. First, we use the experimentally known fact that an electromagnetic wave approximately does not lead to heat production in an insulator through which it moves. This corresponds to the fact that the wave is scattered elastically on the particles constituting the matter. So we must practically have $f_4 = 0$. By means of the field equations we can form the expression

$$\nabla \cdot c(\boldsymbol{E} \times \boldsymbol{H}) + \frac{\partial}{\partial t} \tfrac{1}{2} (\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}) = 0, \tag{6.1}$$

which is consistent with the continuity equation for electromagnetic energy when the energy flux equals

$$\boldsymbol{S} = c(\boldsymbol{E} \times \boldsymbol{H}). \tag{6.2}$$

It is of course true that (6.1) does not unambiguously determine the energy flux to be given by (6.2); for instance, $\boldsymbol{S}$ could in addition contain a term of the form of a curl. But such possibilities are of no interest for our problem.

To determine the momentum components S_{k4}, we make use of the relation $\boldsymbol{S} = c^2\boldsymbol{g}$, whence

$$\boldsymbol{g} = \frac{1}{c}(\boldsymbol{E}\times\boldsymbol{H}). \tag{6.3}$$

The components that we have found up till now constitute a tensor which we shall call[1] $S^A_{\mu\nu}$, with a corresponding force density $\boldsymbol{f}^A$. Our next task is to examine the consequences of this force. Let us therefore consider the simple situation where a plane wave with $\boldsymbol{E} = E_0\boldsymbol{e}_2 \sin(kx-\omega t)$ travels along the x-axis in an isotropic body. We have $k = n\omega/c$, where $n = \sqrt{\varepsilon\mu}$ is the refractive index of the medium. It appears that $f^A_\mu = \delta_{\mu 1}[(n^2-1)/c]$ $(\partial/\partial t)(\boldsymbol{E}\times\boldsymbol{H})_1$, so that there is set up a fluctuating force in the x-direction. This force is rather small; we see that f^A_1 is of the order $(1/c)(n^2-1)(\dot{\boldsymbol{E}}\times\boldsymbol{H})_1$ $= (1/c\mu)(n^2-1)(\varepsilon-1)^{-1}(\dot{\boldsymbol{P}}\times\boldsymbol{B})_1 \approx (1/c)(\dot{\boldsymbol{P}}\times\boldsymbol{B})_1$, which on a microscopical scale (per dipole) corresponds to the magnetic part of the Lorentz force: $(e/c)(\boldsymbol{u}\times\boldsymbol{h})_1$, where $\boldsymbol{h}$ is the microscopical magnetic field and e, $\boldsymbol{u}$ the electric charge and particle velocity, respectively. Now we see that the ratio $u_2/c \ll 1$. In fact, if we accept a simple model with electronic polarization, one dipole per atom, e equal to the electron charge, and put h equal to the macroscopical field strength $B \approx E$ which is set equal to 10 volt/cm, we obtain with optical frequencies $\beta = u/c \approx 10^{-10}$, where $u = |\boldsymbol{u}|$. Such a rough estimation is sufficient to show that quantities proportional to β^2 can be taken to vanish. For instance, since the force on the dipoles in the x-direction is of the order of β times the force on the dipoles in the y-direction, we have also a particle velocity in the x-direction which is $u_1 \approx \beta u_2$. The work performed by f^A_1 per unit time is then $f^A_1 u_1 \approx \beta^2 \times$ (work performed by f^A_2 per unit time) ≈ 0. This is consistent with the result above which also has experimental support: $f^A_4 = 0$.

Let us now introduce a mechanical energy-momentum tensor $U_{\mu\nu}$ such that

$$-\partial_\nu S^A_{\mu\nu} = f^A_\mu = \partial_\nu U_{\mu\nu}. \tag{6.4}$$

In writing this equation, we have already assumed that gravity effects are absent. For instance, if the medium is a fluid then, in the absence of fields,

[1] $S^A_{\mu\nu}$ is equal to Abraham's tensor.

the diagonal components of the stress tensor are equal to the pressure, and the divergence of U_{ik} yields the gravitational force density. But since these effects are of no principal interest here, we shall omit them; hence we interpret the tensor $(S^A_{\mu\nu} + U_{\mu\nu})$ to describe a closed system.

With the plane wave considered the only interesting component of U_{ik} is U_{11}, the effect from the wave on the other components of U_{ik} is zero. The force f_1^A can be thought to act in two ways. (1) It may cause each dipole to fluctuate about a fixed position, the same position as the dipole occupies when the fields are absent. (We ignore thermal motions, which are of no interest here.) On an average, no momentum is then transferred to the dipoles; instead, a kind of small stress is set up. (2) But the effect may also be that a momentum in the x-direction actually results. Since ∂_1 can be replaced by $-(n/c)(\partial/\partial t)$, we obtain from (6.4), with $U_{14} = icg_1^{\text{mech}}$,

$$\frac{\partial}{\partial t}\left(-\frac{n}{c}U_{11} + g_1^{\text{mech}}\right) = f_1^A. \tag{6.5}$$

From this point of view the main effect of f_1^A is to produce a mechanical momentum, so we shall assume the contribution to U_{11} from mechanical stresses to be vanishingly small. Furthermore, the component U_{11} contains also a part $\varrho_m u_1^2$ corresponding to the kinetic energy of the motion in the x-direction, but the quotient $\varrho_m u_1^2/(cg_1^{\text{mech}}) = u_1/c \ll 1$, so that this kinetic part can be neglected. Hence, ignoring the first term in the parenthesis in (6.5), we obtain by means of (6.3) and (1.5 a)

$$g_1^{\text{mech}} = \frac{n^2-1}{c}(\boldsymbol{E}\times\boldsymbol{H})_1 + \text{const}, \tag{6.6}$$

where the constant may depend on E_0.

At this point we cannot get any further by theoretical considerations. We shall therefore seek the remaining information from *experiments* in optics. In this paper we shall consider three experiments which are of importance for our problem; these experiments are mutually in agreement and especially two of them seem to yield sufficient information as to which energy-momentum tensor should be taken as the most convenient. The first experiment—which has immediate application in the present situation—is the Jones-Richards experiment to be described below. The two other experiments are related to the propagation of light in moving media, and will be described later in section 9.

Jones and Richards measured the radiation pressure on a metal vane from an electromagnetic wave passing through a dielectric liquid. Their result is most easily explained by attributing a momentum density $(1/c)$ $\boldsymbol{D} \times \boldsymbol{B}$ to the wave. This behaviour is consistent with assuming the alternative (2) above to be correct and putting the integration constant in (6.6) equal to zero. Thus

$$\boldsymbol{g}^{\text{mech}} = \frac{n^2 - 1}{c} (\boldsymbol{E} \times \boldsymbol{H}). \tag{6.7}$$

We note that (6.6) cannot be supplemented with some initial condition to give an unambiguous result. SOMMERFELD[12], for example, has examined the behaviour of "die Vorläufer", i. e. the incoming field before the stationary state is achieved. The result is that at first the field frequencies are much higher than the atomic frequencies of the medium. Therefore dispersion effects must occur, in contradiction to the assumptions leading to (6.6).

We then turn our attention to the components $U_{4\nu}$. We found above that $f_1^A u_1$ was practically zero, therefore the mechanical energy density $W^{\text{mech}} = -U_{44}$ must also be practically equal to the rest mass density. (The contribution to the energy on account of the force components lying in the yz-plane is already incorporated in S_{44}^A.) The actual equation of motion is

$$\frac{\partial S_1^{\text{mech}}}{\partial x} + \frac{\partial W^{\text{mech}}}{\partial t} = 0, \tag{6.8}$$

where S_1^{mech} denotes the flow of mechanical energy in the x-direction. According to the principle of inertia of energy we can put $\boldsymbol{S}^{\text{mech}} = c^2 \boldsymbol{g}^{\text{mech}}$, where $\boldsymbol{g}^{\text{mech}}$ is given by (6.7). $\boldsymbol{S}^{\text{mech}}$ corresponds to a very small motion of dipoles; with the simple model above we found that $u_1 \approx 10^{-10}$ cm/s and because of the elastic coupling to the atoms the motion will be even smaller.

The kinetic energy on account of this motion is of course practically zero, but yet a finite energy transport is achieved by the great *rest mass*. As the wave proceeds through the body, new domains of matter are continuously being excited; and when the wave has passed, the dipoles have been displaced by a small amount in the x-direction.

Now, after having interpreted the components of $U_{\mu\nu}$, we introduce the quantities $\Theta_{\mu\nu}$ defined by

$$\Theta_{i\nu} = U_{i\nu}, \ \Theta_{4\nu} = 0, \ (i = 1,2,3; \ \nu = 1-4). \tag{6.9}$$

Then

$$-\partial_\nu (S_{\mu\nu}^A + \Theta_{\mu\nu}) = -\partial_\nu S_{\mu\nu}^M = 0, \tag{6.10}$$

where $S^M_{\mu\nu}$ is Minkowski's tensor, which accordingly describes the propagation of the total travelling system, both the electromagnetic field and the mechanical excitation caused by the field. The small displacement of matter and the rest energy itself are ignored in this context.

Concerning $\boldsymbol{g}^{\mathrm{mech}}$ given by (6.7) we note that this mechanical quantity is expressed chiefly by electromagnetic ones. This is a characteristic feature of the phenomenological theory, and similar things are also found for instance in the expression for the electrostatic field energy density in an isotropic medium where, besides the pure field part $\frac{1}{2}E^2$, there appears an amount of internal energy in the medium, which is written as $\frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{P} = \frac{1}{2}(\varepsilon-1)E^2$.

The Jones-Richards experiment[13]

We shall now consider the experiment to which we referred above in order to find the result (6.7). In 1951, R. V. Jones[13a] first reported in a short note a measurement of the radiation pressure in various dielectric fluids, and later, in 1954, R. V. Jones and J. C. S. Richards[13b] gave an extensive report of the final experiment. We find that this excellent experiment clearly demonstrates that it is most simple and convenient to ascribe a momentum density $(1/c)(\boldsymbol{D}\times\boldsymbol{B})$ to an optical wave travelling through a refracting fluid. The experimental arrangement was the following: A ray of light passed through a glasswindow into a dielectric liquid and was reflected in the opposite direction by a metal vane immersed in the liquid. (Actually, the authors used two rays of light which were falling asymmetrically on the vane, and the vane was mounted on a torsional suspension.) The ratio between the pressure on the vane when it was immersed in the liquid and the pressure on the vane when it was surrounded by air was measured. This ratio was found to be equal, the external conditions also being equal, to the refractive index of the fluid. Let us apply a simple theoretical argument and first consider the divergence-free Minkowski's tensor with momentum density equal to $g = (1/c)DB = n^2S/c^2$. The symbols are referring to the incoming wave in the liquid. The momentum transferred to a unit surface of the vane per unit time is thus $p_n = (1/c)nS(1+R)$, where R is the reflectivity of the vane. Dividing by the vacuum (air) pressure $p_0 = (1/c)S_0(1+R_0)$ and assuming $S_0 = S$ and $R_0 = R$, we find indeed the simple formula $p_n/p_0 = n$. (See also the analysis by G. Rosenberg[8].) It is evident that a number of corrections are called for in this formula, owing to the fact that the external conditions in reality are varying with n. For instance, although the intensity of the radiation source (which is outside the container) is kept constant in the experiment, the intensity S will depend on the refractive index in a way which may be described by means of Fresnel's formulas: The electric field E of the incoming wave in the liquid is related to the electric field E_g of the incoming wave in the glass by $E = 2E_g/(1+n/n_g)$, where n_g is the refractive index of the glass and μ is put equal to unity. Hence $p_n/p_0 = nS/S_0 = (1+n_g)^2(1+n_g/n)^{-2}$ which, in the case of a typical fluid, amounts to a correction of approximately 4 % with respect to the simple formula quoted above.

Apart from this correction, Jones and Richards carefully took into account corrections arising from other effects, such as absorption in the liquid, multiple reflections at the vane and the window, and dependence of the reflectivity R on the refractive index of the fluid. Unwanted effects from convective forces in the liquid were eliminated experimentally by means of a chopping technique. After these various secondary effects had been compensated for, the agreement between theory and experiment was found to hold within approximately 1 % for all the six various liquids investigated. This agreement is remarkable, in consideration of the small effects involved (the mechanical couple measured was of the order of 10^{-6} dyne cm).

If now in the calculation above we had inserted the expression $g = (1/c)$ EH for the momentum density, we would have got a factor $1/n^2$ different and hence disagreement with the observed data. This does not mean, however, that Minkowski's momentum density is correct and all other alternatives wrong, for the calculation above applies only to the case of a *divergence-free* tensor. The experimental result does not prevent us from using an energy-momentum tensor with a non-vanishing force density such that the effect from the force is to be added to the effect considered above. But for a divergence-free tensor, the experiment supports Minkowski's expression.

Anisotropic matter

This situation is analogous to the preceding one so we shall not go into detailed considerations. We may *choose* the stress tensor to be given by (1.5a) also in this case, in accordance with the dipole model from section 3. By using the same argument as before, we find that the energy flux and momentum density of the field are given by (6.2) and (6.3). The four-force density f'_μ derived from this preliminary energy-momentum tensor is given by $\boldsymbol{f}' = (1/c)(\partial/\partial t)(\boldsymbol{D}\times\boldsymbol{B}-\boldsymbol{E}\times\boldsymbol{H})$, $f'_4 = 0$, when no charges or currents are present. Then we suppose that this force excites a mechanical momentum density $(1/c)(\boldsymbol{D} \times \boldsymbol{B} - \boldsymbol{E} \times \boldsymbol{H})$ which travels together with the field. Including this quantity in the energy-momentum tensor, we obtain finally Minkowski's tensor as given by (1.5). That $S^M_{k4} \neq S^M_{4k}$ corresponds to the fact that the small motion of matter particles is not taken into account, while the asymmetry of the spatial components S^M_{ik} is connected with torques.

7. On the Microscopical Method of Approach

Even though we are concerned mainly with the phenomenological theory and in the preceding section employed an intermediate method, we shall here mention some papers where more or less microscopical theories have been developed.

First, we refer to the treatment of TANG and MEIXNER(14). This method is not purely microscopical, and the main idea is rather similar to that we presented above. The authors make use of the total energy momentum tensor written in a form given earlier by KLUITENBERG and DE GROOT(15), and examine the excitation of matter set up by a plane electromagnetic wave travelling in a fluid. From the differential conservation equations they obtain an expression for the velocity variations and hence evaluate the total energy-momentum tensor in a form where the oscillating terms are shown explicitly. On a time average the formal results are compatible with the results we earlier obtained. We should perhaps point out, however, that in spite of the formal completeness of the method one should in addition use experimental results to get information about the average velocity of matter in the original rest frame. For instance, in the frame where the constituent particles have no mean motion, one ends up with $S^A_{\mu\nu}$ plus the tensor corresponding to the rest mass properties of the medium as the total one.

Next, we shall take up a question which has led to one of the strongest arguments in favour of a symmetrical tensor: The macroscopical tensor $S_{\mu\nu}$ should be derivable from the corresponding symmetric, microscopical tensor $s_{\mu\nu}$ by averaging over appropriate regions in space-time, and should thus maintain its symmetry property. This argument was originally given by ABRAHAM(16), and his view seems to have been supported by several physicists (i. e. LANDAU and LIFSHITZ(4), PAULI(17)).

But it can be seen that averaging procedures do not make difficulties for Minkowski's theory. Consider a limited electromagnetic field within an insulator; by averaging over space-time elements, we obtain for the torque density in component form $-\overline{x_i \partial_\nu s_{k\nu}} + \overline{x_k \partial_\nu s_{i\nu}}$. Comparing with the corresponding torque calculated from the macroscopical tensor, we get

$$-\bar{x}_i \partial_\nu S_{k\nu} + \bar{x}_k \partial_\nu S_{i\nu} + S_{ik} - S_{ki} = -\overline{x_i \partial_\nu s_{k\nu}} + \overline{x_k \partial_\nu s_{i\nu}}. \tag{7.1}$$

Now introducing the dipole model in charge-free homogeneous regions of the anisotropic body, (7.1) reduces to

$$S_{ik} - S_{ki} = -\overline{x_i \partial_\nu s_{k\nu}} + \overline{x_k \partial_\nu s_{i\nu}}. \tag{7.2}$$

The right hand side of (7.2) is not necessarily equal to zero, therefore S_{ik} is not equal to S_{ki} in general. This result is what we might have expected; while it is sufficient to regard the macroscopical tensor to be given by the averaged microscopical one in regard to *linear* quantities (forces), this consideration is insufficient in regard to *second order* quantities such as torques. We have $f_i = -\partial_\nu S_{i\nu} = -\partial_\nu \overline{s_{i\nu}}$, but $\overline{s_{i\nu}}$ cannot express that the microscopical forces act at different points within a dipole. However, $S_{\mu\nu}$ must take into account the macroscopical effects arising also from this fact.

The above reasoning is mainly the same as that carried out by Ig. TAMM (see ref. 1, § 75).

Then we shall consider to some extent the recent series of papers by DE GROOT and SUTTORP(18). These papers represent presumably the most extensive microscopical treatment of the problem that has appeared. The advantage of a purely microscopical method is that one obtains expressions for the *total* energy-momentum tensor, the sum of the electromagnetic and the mechanical part. DE GROOT and SUTTORP give two expressions for the electromagnetic energy-momentum tensor, both of which are different from Minkowski's tensor. They claim that Minkowski's tensor (and also Abraham's tensor) cannot be justified from a microscopical point of view. Their first proposal, obtained by means of statistical arguments, reads in the momentary rest system of matter, if the body is a fluid,

$$S_{ik} = -E_i D_k - H_i B_k + \delta_{ik}(\tfrac{1}{2} E^2 + \tfrac{1}{2} B^2 - \boldsymbol{M} \cdot \boldsymbol{B}) \tag{7.3 a}$$

$$S_{4i} = S_{i4} = i(\boldsymbol{E} \times \boldsymbol{H})_i \tag{7.3 b}$$

$$S_{44} = -\tfrac{1}{2}(E^2 + B^2), \tag{7.3 c}$$

where these terms have been extracted from the expression for the total tensor. But we have to point out that this is not primarily a *derivation* of the electromagnetic tensor, it is a *choice*. There is no a priori reason to take out just these terms and consider them as constituting the electromagnetic tensor, even though it seems to be the simplest choice from a formal point of view. For the macroscopical fields are contained also in the remaining terms of the total tensor, although they are there mixed up with mechanical quantities. This ambiguity of splitting is inherent in any microscopical theory. DE GROOT and SUTTORP claim that in a macroscopical treatment, in which the material tensor is not determined, the problem is to a large extent undetermined. We agree that there is an ambiguity present in the macroscopical theory—the problem is to some extent a matter of convenience—but we must point out that this ambiguity is not removed upon transition to the microscopical theory.

DE GROOT and SUTTORP also employ thermodynamic methods and give another form for the electromagnetic tensor which includes the whole interaction between field and matter, i. e., it is equal to the total tensor, minus the mechanical tensor in the absence of macroscopical fields. This tensor is interesting since it is closely connected with the result we obtained macroscopically. (The stress tensor obtained in section 3 was based on the free energy (3.1) in the electrostatic case, and this quantity certainly contains the whole interaction between field and matter since it is equal to the work exerted in building up the field.) Actually, this result is compatible with Minkowski's tensor, if one ignores the dependence of the material constants on the density and temperature, as we have done in our investigation, and one employs our former interpretation concerning the moving dipoles in K^0. For in the frame where the matter has no mean motion, their tensor agrees with Minkowski's tensor, except for terms involving gradients of the material constants, and except for the momentum components which are given as $S_{i4} = i(\boldsymbol{E} \times \boldsymbol{H})_i$. If we then go over to the original rest frame K^0 and add the contribution to the momentum from the small motion of the constituent particles in K^0, we obtain Minkowski's tensor. The corresponding contribution to the energy flux is included in the mechanical tensor.

Summing up these remarks, we think that the microscopical theory, involving a derivation of the total energy-momentum tensor, is an interesting and very complete treatment of the problem. Both the macroscopical and the microscopical method imply certain ambiguities, the first one because the mechanical tensor is not determined in this way, the second one because the splitting of the total tensor is not unique. However, if the task is to determine the electromagnetic tensor which is most convenient and therefore ought to be used, we think that the macroscopical method is both effective and by far the simplest method, if one in addition takes into account the experimental results.

Finally, we mention some microscopical treatments in which only the field part of the total energy-momentum tensor has been derived. H. OTT[(19)]

made an attempt to deduce the macroscopical electromagnetic tensor (assumed to be symmetric) by averaging over the microscopical quantities and imposing the subsidiary condition that, for an optical field, the four-component of force f_4 should be zero. Further, DÄLLENBACH[20] made use of the electron theory to give a covariant derivation of the electromagnetic tensor. He obtained Minkowski's tensor as the result. These different results reflect characteristic ambiguities that are encountered, and we shall not go into further details.

IV. Further Developments, Connected with Relativity Theory

This chapter contains extensions and applications of results that have been obtained up till now. In particular, we shall be interested to demonstrate explicitly the characteristic features that are encountered when Minkowski's tensor is used. Thus we shall consider both specific examples and more deductive procedures which are intimately connected with Minkowski's tensor. These topics have been rather extensively studied in the literature. In this chapter we consider isotropic media only.

8. The Canonical Energy-Momentum Tensor

The Lagrangian and the Hamiltonian formalisms in special relativity are frequently used in order to find the energy-momentum tensor of some system. Let us apply this kind of method to the situation where an isotropic and homogeneous medium, containing a radiation field, is moving with the uniform four-velocity V_μ. We may start from Noether's theorem, which here can be written

$$\frac{\partial}{\partial x_\nu}\left[\left(L\delta_{\mu\nu}-\frac{\partial L}{\partial A_{\alpha,\nu}}A_{\alpha,\mu}\right)\delta x_\mu+\frac{\partial L}{\partial A_{\alpha,\nu}}\delta A_\alpha\right]+\frac{\partial L}{\partial V_\mu}\delta V_\mu = 0. \qquad (8.1)$$

Here A_μ is the electromagnetic four-potential, and $A_{\alpha,\nu} = \partial_\nu A_\alpha$. Further, L is the Lagrangian density, which we choose as

$$L = -\tfrac{1}{4}F_{\mu\nu}H_{\mu\nu} = -\frac{1}{4\mu}F_{\mu\nu}F_{\mu\nu}+\frac{\varkappa}{2\mu}F_\mu F_\mu . \qquad (8.2)$$

$H_{\mu\nu}$ is the tensor defined in section 1; the covariant relation between $H_{\mu\nu}$ and $F_{\mu\nu}$ is

$$\mu H_{\mu\nu} = F_{\mu\nu} + \varkappa (F_\nu V_\mu - F_\mu V_\nu), \tag{8.3}$$

where $\varkappa = (\varepsilon\mu - 1)/c^2, F_\mu = F_{\mu\nu} V_\nu$. It can readily be verified that the variational equations

$$\frac{\partial L}{\partial A_\mu} - \frac{\partial}{\partial x_\nu} \frac{\partial L}{\partial A_{\mu,\nu}} = 0 \tag{8.4}$$

with L inserted from (8.2) lead to Maxwell's equations. In the derivation of (8.1), eqs. (8.4) have been used. For a derivation of Noether's theorem in general see, for instance, the review paper by E. L. HILL(21).

The δ-quantities in (8.1) refer to infinitesimal symmetry transformations of coordinates and dependent variables, i. e. the field equations must be unchanged in form under the transformations. Employing the infinitesimal translation in four-space $x'_\mu = x_\mu + \delta x_\mu, \delta x_\mu =$ const as a symmetry transformation, we obtain from (8.1), since δA_μ and δV_μ vanish

$$\partial_\nu S_{\mu\nu}^{\mathrm{can}} = 0, \tag{8.5 a}$$

where

$$S_{\mu\nu}^{\mathrm{can}} = L\delta_{\mu\nu} - \frac{\partial L}{\partial A_{\alpha,\nu}} A_{\alpha,\mu} \tag{8.5 b}$$

is the canonical energy-momentum tensor. By means of (8.2) we then find

$$S_{\mu\nu}^{\mathrm{can}} = H_{\nu\alpha} A_{\alpha,\mu} - \tfrac{1}{4} \delta_{\mu\nu} F_{\alpha\beta} H_{\alpha\beta}. \tag{8.6}$$

This tensor is neither symmetric nor gauge-invariant. In order to eliminate gauge-dependent quantities we may add $H_{\alpha\nu} A_{\mu,\alpha}$ on the right hand side of (8.6), whereby we obtain Minkowski's tensor. The additional term is divergence-free, and does not influence the conserved four-momentum obtained from $S_{\mu\nu}^{\mathrm{can}}$. (When $\varkappa = 0$ the electromagnetic field becomes a *closed* system, and in that case the additional term may be found by means of the well known field theoretical symmetrization procedure, due originally to BELINFANTE(22) and ROSENFELD(23).)

It is thus apparent that Minkowski's tensor readily adjusts itself to the canonical procedure. We have to emphasize, however, that the foregoing procedure does not determine Minkowski's tensor uniquely. One of the reasons is that the Lagrangian density (8.2) corresponds to a non-closed system and thus we have, from a formal point of view, no initial information as to whether the four-force density vanishes or not. *If* we demand that the force density shall vanish, then Minkowski's tensor is the simplest result emerging from the formalism. But this tensor is still not determined uniquely, since there is no a priori reason that only the field quantities be present in the electromagnetic tensor. Terms involving the material constants ε and μ and the four-velocity V_μ may be present, and still the tensor may be divergence-free.

We mention that some interest has been given to the problem of how to make use of the phenomenological Lagrangian methods sketched above and then construct the Lagrangian and energy-momentum tensor for the *total* system, matter plus field. We may refer to a paper by SCHMUTZER(24), who as a result claimed Minkowski's tensor to be preferred for the field. It is obvious, however, that the same kind of ambiguity in the formalism is encountered here as in the microscopical theory we remarked upon in section 7: One does not know which division of the total tensor into electromagnetic and mechanical parts should be chosen. One ought to have some information from experiments in simple physical cases in order to make a convenient choice.

Finally we mention that the problem of constructing the total energy-momentum tensor is encountered also in magnetohydrodynamics, a field that seems to have attracted considerable interest during the last years. These works are carried out on a phenomenological level. Now the mechanical energy-momentum tensor for the fluid, in the absence of a field, is symmetric. If Minkowski's tensor is chosen for the field, as is often the case, one then has to add an "interaction" tensor in order to make the total tensor symmetric. See the papers by PICHON(25), PHAM MAU QUAN(26) and RANCOITA(27).

9. Transformation of the Velocity of the Energy in a Light Wave. Two experiments

Consider a plane light wave within an isotropic and homogeneous insulator moving with the uniform four-velocity V_μ in the reference frame K. One defines the so-called ray velocity $\boldsymbol{u}$ as the velocity of propagation of the light energy. It is known that, similarly as in the case of an anisotropic body at rest, one has to distinguish between the ray velocity and the phase velocity. For an electromagnetic field in the vacuum, the ray velocity and phase velocity become in general equal. They are equal also in the presence of an isotropic medium in the special case when the medium is at rest, or, more generally, when the ray is parallel to the direction of the motion of the medium.

It is shown in Møller's book[(1)] that the ray velocity transforms like the velocity of a material particle. He starts with the following equation for the wave front of a spherical wave in K^0 being emitted from the origin at the time $t^0 = 0$:

$$r^{02} - \frac{c^2}{n^2} t^{02} = 0. \tag{9.1}$$

Further—and that is a crucial point—the corresponding equation for the wave front in K is found by means of the usual *point transformations* of each term in (9.1). That means that the world lines of the propagating wave are assumed to remain invariant in four-space upon a Lorentz transformation. From theoretical considerations there seems to be no cogent reason that nature really should conform to this assumption (it has sometimes been claimed that if a particle travels in the light in one inertial frame it will stay in the light also in another frame, but obviously this can be true only *if* the ray velocity transforms like the particle velocity). However, if we again invoke *experimental* results, such as those obtained in the Fizeau experiment described below, we find that the considered transformation property of the ray velocity actually is verified in simple physical situations. We shall see that this circumstance establishes a simple criterion which an electromagnetic energy-momentum tensor ought to fulfil, in order to be convenient.

Let $S_{\mu\nu}$ be an electromagnetic tensor which shall describe the travelling wave. Since $\boldsymbol{u}$ is defined as the velocity of propagation of the wave energy, we have $u_i = icS_{4i}/S_{44} = S_i/W$. This velocity transforms like a particle velocity if and only if the quantities

$$U_\mu = \left(\frac{\boldsymbol{u}}{\sqrt{1-u^2/c^2}}, \quad \frac{ic}{\sqrt{1-u^2/c^2}} \right) \tag{9.2}$$

constitute a four-vector. By performing an infinitesimal Lorentz transformation $x'_\mu = x_\mu + \omega_{\mu\nu} x_\nu$ between two inertial frames K and K', Møller[(1)] has shown that U_μ transforms like a four-vector between these systems when

$$R_{\mu\nu} \equiv S_{\mu\nu} + \frac{1}{c^2} S_{\mu\alpha} U_\alpha U_\nu \tag{9.3}$$

vanishes in K. Since a finite Lorentz transformation may be composed of infinitesimal transformations, the equation $R_{\mu\nu} = 0$ is a general condition that $S_{\mu\nu}$ must satisfy in order that $\boldsymbol{u}$ shall have the required transformation property.

It is easily seen that it is sufficient to examine $R^0_{\mu\nu}$ in K^0.

Møller shows that $R^0_{\mu\nu} = 0$ with Minkowski's tensor, when the most general solution of the field equations representing a plane wave in K^0 is inserted. This feature means that Minkowski's tensor gives an adequate description of the velocity of the energy in a light wave in any inertial system.

Similar conclusions have been drawn by several authors. The subject was first treated long ago by Scheye[28]. It was elaborated by von Laue and published in a paper in 1950[29]. Another treatment was worked out, independently and almost simultaneously, by Møller, and published in his book in 1952[1]. We may refer also to a paper by Schöpf[30]. It has been shown by Manarini[31] that $\boldsymbol{u}$ given by Minkowski's tensor transforms like a particle velocity also within anisotropic media.

Fizeau's experiment

Assume that the ray travels parallel to the direction of motion of the medium. By using Minkowski's tensor, or simply by transforming the ray velocity $\boldsymbol{u}$, we find in K, to the first order in v/c,

$$u = \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right), \tag{9.4}$$

where the expression in the parenthesis is Fresnel's dragging coefficient.

Fizeau checked the formula (9.4) experimentally. He used a two-beam interferometer with moving water in the beam path. The phase difference between the two beams was measured and was found to be in agreement with the result predicted on the basis of (9.4). Zeeman even verified the dispersion correction term to the formula (9.4). For a more detailed description of the experiment, and for references to the original literature, see § 8 in Møller's book[1].

[*Note added in the proof:* It has recently come to our attention that this kind of experiment has recently been repeated by W. M. Macek, J. R. Schneider, R. M. Salamon, Journ. Appl. Phys. **35**, 2556 (1964). The authors made use of a ring laser in order to measure the phase difference between the waves, thereby improving the sensitivity by several orders of magnitude. The dragging coefficient was measured in both a solid, a gaseous medium and a liquid, and especially in the two first cases the agreement with the expression $(1 - 1/n^2)$ was found to be good.]

In a paper[(14)] which we also referred to in section 7, TANG and MEIXNER constructed an expression for the total energy-momentum tensor and also examined the transformation criterion of VON LAUE and MØLLER in connection with a physical interpretation of the various terms in this tensor. Recently, DE GROOT and SUTTORP[(18)] claimed that TANG and MEIXNER in this paper actually invalidated the transformation criterion. We cannot, however, agree with this statement. At least in the simple situation considered here the mentioned transformation property of the ray velocity $\boldsymbol{u}$ is verified *experimentally*; further, the relation $\boldsymbol{u} = \boldsymbol{S}/W$ ought to be valid for an electromagnetic tensor which shall describe the total light wave.

A Sagnac-type experiment

In a recent paper HEER, LITTLE and BUPP[(32)] reported an experiment involving the propagation of light through dielectric media in an accelerated system of reference. This is thus a kind of generalization of the Fizeau experiment, which involved inertial systems only. Let us sketch some important features of this new experiment.

The apparatus is a triangular ring laser as shown in Fig. 1. L is a gas laser which gives rise to two travelling electromagnetic waves in the cavity, one circulating clockwise and the other counterclockwise. When the system is at rest the photon frequencies in the two wave modes are equal. Then imagine that the cavity is set into rotation with an angular velocity Ω, such that the direction of Ω is perpendicular to the cavity plane shown in the figure. The photon frequencies of the two beams now become different from each other; the beams interfere to produce beats which are counted at the detector D. This rotation-dependent frequency shift is called the Sagnac effect (see the review paper by POST[(33)]). If a dielectric medium F is placed in the light path, the effect will depend on the geometry of the medium and on the velocity of light inside it, and will hence be connected with the electromagnetic energy-momentum tensor. This connection can be expressed in mathematical form as follows[(34)]. The energy density W for one of the modes in the cavity frame is related to the energy density W^0 for this mode in an instanteneous inertial rest frame by

$$W = W^0 + \frac{1}{c}\Omega \cdot [\boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{H})]. \tag{9.5}$$

Only effects to the first order in Ω are investigated, so that the fields in (9.5) may be evaluated for $\Omega = 0$. Within this approximation the integral $H =$

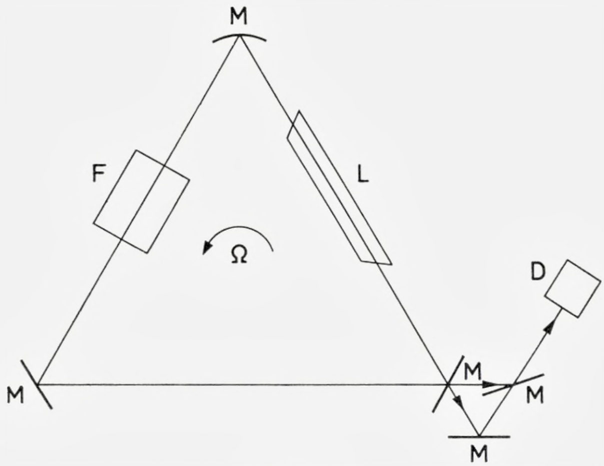

Fig. 1.

$\int W dV$, taken over the volume of the field, is a conserved quantity. Further, the integral of W^0 over the volume is the same for the two modes, so we obtain for the relative frequency shift

$$\left.\begin{aligned} \Delta\nu/\nu &= \Delta H/H = \\ &= (4/c)[\Omega\cdot\int \boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{H})dV]\left[\int(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{H}\cdot\boldsymbol{B})dV\right]^{-1}. \end{aligned}\right\} \quad (9.6)$$

Considering the beam as a plane wave with a small cross section, we obtain from (9.6)

$$\Delta\nu/\nu = (4\Omega A/c)\left[\int(n+\nu dn/d\nu)dl\right]^{-1}, \quad (9.7)$$

where A is the area enclosed by the light path and dl the line element along the light path. In (9.7) also the correction from the dispersion has been included. The frequency shift $\Delta\nu$ is simply equal to the number of beats counted per unit time.

The material medium F in the beam path was chosen as pairs of quartz plates at anti-parallel Brewster angles. The value of the integral in (9.7) could thus be varied by varying the number of pairs. In order to eliminate the influence from the rotation of the Earth, one had to take the mean of the results obtained by rotating the cavity in the clockwise and in the counterclockwise direction. The agreement between the observed data and the results obtained on the basis of (9.7) was excellent.

As a conclusion, we find that both the JONES-RICHARDS experiment considered in section 6 and the two experiments considered in this section are explained on the basis of Minkowski's tensor in a very simple way[1]. And this is the main reason why we consider Minkowski's tensor to be convenient for the description of optical phenomena.

10. Negative Energy. Remarks on the Čerenkov Effect

Negative energy

By making use of Minkowski's tensor we find that the electromagnetic field energy becomes negative under certain circumstances, and this fact has caused difficulties for the acceptance of this tensor. We shall show that such a behaviour is a consequence of the way in which the covariant theory is constructed.

Consider a plane electromagnetic wave which moves along the x-axis within an isotropic and homogeneous insulator with index of refraction denoted by n. If W^0 is the field energy density in the rest frame K^0 of the body and $v = v_1 = c\beta$ the velocity of K^0 with respect to an inertial frame K, we find that Minkowski's energy density in K is

$$W^M = \gamma^2(1+n\beta)(1+\beta/n)W^0. \tag{10.1}$$

From this expression it follows that $W^M < 0$ when $\beta < -(1/n)$.

This feature is, however, connected with the fact that the rest mass quantities of the medium have been excluded from $S^M_{\mu\nu}$. For the tensor $\Theta_{\mu\nu}$ introduced in (6.9) has the only non-vanishing component $\Theta^0_{14} = icg_1^{\text{mech}\,0} = i[(n^2-1)/n]W^0$ in K^0, which means that in K

$$-\Theta_{44} = \beta\gamma^2[(n^2-1)/n]W^0. \tag{10.2}$$

Hence, the contribution to the energy density is negative when β is negative.

For illustration, let us consider the following analogous situation from mechanics: A material particle with four-momentum $p_\mu = (\boldsymbol{p}, iE/c)$ moves uniformly along the x-axis and is considered in two frames K and K', where K' moves with the velocity v with respect to K. Then $E = \gamma(vp'+E')$ and is of course positive; but by ignoring E', we obtain $E < 0$ when $v < 0$, provided $p' > 0$. This is the same effect as encountered above. For a material particle ignoring E' is of course impossible, since we know the relations between $\boldsymbol{p}, E$ and $\boldsymbol{p}', E'$ from the Lorentz transformation and the principle

[1] As we shall see later, the two first of these experiments represent a more critical test than the last one.

of covariance (cf. § 26 in MØLLER's book[1]), and thus we have only to find that combination $p_\mu = (\boldsymbol{p}, iE/c)$ which makes up a four-vector. But the covariant phenomenological electrodynamics is achieved by *choosing* appropriate four-vectors and tensors which in K^0 are coincident with already established quantities, such as the four-force density. In the picture corresponding to Minkowski's tensor we include the mechanical momentum density $\boldsymbol{g}^{\text{mech } 0}$ into the electromagnetic tensor, but not the quantities $\boldsymbol{S}^{\text{mech } 0}$ and $W^{\text{mech } 0}$. By requiring covariance of this picture, we obtain a space-like, total four-momentum G_μ^M of the field. Therefore, by means of proper Lorentz transformations, we can find inertial frames where the field energy is negative.

The Čerenkov effect

This effect offers an interesting application of Minkowski's theory. We shall suppose that an electron moves along the x-axis with a uniform velocity which in K^0 is larger than c/n, the light velocity in the medium. And we shall consider the process in the inertial frame K where the electron is at rest. In this frame we find that the fields are stationary, and that $\boldsymbol{H} = 0$[35]. Let us then integrate the differential conservation laws over a volume which contains the electron and which is enclosed by a cylindric surface S of small radius and infinite length such that the axis of the cylinder coincides with the x-axis. As $\boldsymbol{H} = 0$, the energy flow through S vanishes; the field energy does not change, and the work exerted by the electromagnetic force on the electron is zero.

Then examine the momentum balance. Unlike the energy flow the momentum flow is different from zero[35], and the momentum transport through S corresponds to a force on the electron in K. This is again a characteristic consequence of the peculiar construction of Minkowski's momentum density in K^0. The momentum balance in K reads

$$\int S_{ik}^M n_k dS = -\int f_i^M dV, \tag{10.3}$$

where the force components on the right hand side are readily obtained in K by transforming Minkowski's force from K^0.

We shall return to this situation in the next paper, in connection with Abraham's tensor.

11. Angular Momentum

We begin with some general remarks in connection with the application of Noether's theorem as given by (8.1). By employing the infinitesimal Lorentz transformation $\delta x_\mu = \omega_{\mu\nu} x_\nu$ as a symmetry transformation in (8.1), we obtain (isotropic media assumed)

$$\frac{\partial M_{\mu\nu\sigma}}{\partial x_\sigma} + \frac{\partial L}{\partial V_\alpha} I^{\alpha\beta}_{\nu\mu} A_\beta = 0, \tag{11.1}$$

where

$$M_{\mu\nu\sigma} = x_\mu S^{\mathrm{can}}_{\nu\sigma} - x_\nu S^{\mathrm{can}}_{\mu\sigma} + \frac{\partial L}{\partial A_{\alpha,\,\sigma}} I^{\alpha\beta}_{\nu\mu} A_\beta \tag{11.2}$$

and

$$I^{\alpha\beta}_{\nu\mu} = \delta_{\nu\alpha}\delta_{\mu\beta} - \delta_{\nu\beta}\delta_{\mu\alpha}. \tag{11.3}$$

$S^{\mathrm{can}}_{\mu\nu}$ is given by (8.5b).

If we interpret $M_{\mu\nu\sigma}$ to be connected with the field angular momentum $M_{\mu\nu}$ by

$$M_{\mu\nu} = \frac{1}{ic}\int M_{\mu\nu 4}\, dV, \tag{11.4}$$

then it can be easily verified that (11.4) is equivalent to $M^M_{\mu\nu}$ obtained from (1.6) with Minkowski's tensor inserted.

From (11.2) we obtain a coordinate-dependent part of angular momentum

$$L^M_{ik} = \int (x_i g^{\mathrm{can}}_k - x_k g^{\mathrm{can}}_i)\, dV = \frac{i}{c}\int \frac{\partial L}{\partial A_{\alpha,\,4}} (x_i \partial_k - x_k \partial_i) A_\alpha\, dV \tag{11.5}$$

and a coordinate-independent part

$$\Sigma^M_{ik} = \frac{1}{ic}\int \frac{\partial L}{\partial A_{\alpha,\,4}} I^{\alpha\beta}_{ki} A_\beta\, dV. \tag{11.6}$$

Inserting L from (8.2), we find

$$L^M_{ik} = \frac{1}{c}\int \boldsymbol{D}\cdot(x_i\partial_k - x_k\partial_i)\boldsymbol{A}\, dV \tag{11.7}$$

$$\Sigma^M_{ik} = \frac{1}{c}\int (D_i A_k - D_k A_i)\, dV. \tag{11.8}$$

Let us then apply the theory to the physical situation in which a plane, monochromatic wave with wave vector $\boldsymbol{k}$ travels within a homogeneous and isotropic insulator moving with the velocity v in the x-direction. A proper discussion of the expressions (11.7) and (11.8) ought to be made in a quantal treatment, but the following general remarks may be made.

As indicated in (1.7), the quantities M_{ik}^{M} are in general not conserved. It can be shown in the present case that this non-conservation is due to the part L_{ik}^{M}, while the contribution from $\sum_{ik}^{M}$ fluctuates away.

It is known that for an electromagnetic field in the vacuum we can in the Coulomb gauge ($A_4 = 0$) interpret (11.7) as the orbital angular momentum, since this part is independent of the polarization of the photons. Similarly, we obtain for $\varkappa = (n^2-1)/c^2 > 0$ that the constant part of L_{ik}^{M} is polarization independent if we use the gauge in K^0 in which $A_4^0 = 0$. If $k_l = 0$ ($l = 2,3$) or $v = 0$, then all quantities M_{ik}^{M}, L_{ik}^{M} and $\sum_{ik}^{M}$ are conserved. In this case L_{ik}^{M} is polarization independent and is thus interpreted as orbital angular momentum, while $\sum_{ik}^{M}$ is interpreted as the spin part.

We can verify that $\sum_{\mu\nu}^{M}$ is not a tensor, except in the special case $k_l = 0$ when the total angular momentum also is a tensor. In an electromagnetic field in the vacuum $\sum_{\mu\nu}$ is a tensor only when $k_l = 0$; however, when $\varkappa = 0$, $M_{\mu\nu}$ is a tensor.

12. Centre of Mass

Consider in K^0 a bounded radiation field, whose interior domain can be taken as a part of a plane monochromatic wave with wave vector $\boldsymbol{k}^0$. Only in a small boundary layer the fields are assumed not to obey the usual plane wave relations, and this boundary layer is further assumed to contain negligible field energy or momentum.

Then let K^0 move with respect to K with the velocity v along the x-axis, and examine the behaviour of the centre of mass in K with coordinates $\boldsymbol{X}(K)$. Taking into account that the field is bounded and that the total field energy $\mathscr{H}^M$ is conserved, we find

$$\frac{d}{dt}X_i^M(K) = \frac{d}{dt}\left(\frac{1}{\mathscr{H}^M}\int x_i W^M dV\right) = \frac{1}{\mathscr{H}^M}\int S_i^M dV. \tag{12.1}$$

Since the field is homogeneous,

$$d/dt X_i^M(K) = S_i^M/W^M = u_i, \tag{12.2}$$

where u_i has been found to transform like the component of a particle velocity. Strictly speaking, Møller's mathematical treatment referred to in section 9 was based on a point transformation, while (12.2) in general refers to different space-time points in two reference frames; however, this does not matter since $\boldsymbol{u}$ is constant along a world line.

We obtain then for the wave under consideration

$$\frac{d}{dt} X_1^M(K) = c\frac{n\beta + k_1^0/k^0}{n + \beta k_1^0/k^0} \tag{12.3a}$$

$$\frac{d}{dt} X_l^M(K) = c\frac{k_l^0/k^0}{\gamma(n + \beta k_1^0/k^0)} \quad (l = 2,3). \tag{12.3b}$$

Here $k^0 = |\boldsymbol{k}^0|$. When $k_1^0 = k^0$, $\beta = -1/n$, K is identical with the rest system K^* of the wave, wherein Poynting's vector and the energy density both vanish in such a way that the quotient (12.3a) also vanishes. If $k_1^0 = k^0$, $\beta < -1/n$, then $S_1^M > 0$, $W^M < 0$, $dX_1^M(K)/dt < 0$.

Investigation of the various mass centres

For a physical system in general, it is known that the different centres of mass we obtain by varying the reference frames K, do not necessarily coincide when considered simultaneously in one frame. We refer to a paper by Møller[36], in which it was shown that different positions may occur in the case of a *closed* system possessing angular momentum in its own rest frame (see also ref. 1). Such a closed system is in many ways similar to our radiation field, so that we wish to study this point. To avoid complicated notation, the superscript M shall be omitted in the following.

Since the rest frame K^0 plays a distinguished role we may call the centre of mass $\boldsymbol{X}(K^0)$ in this frame the proper centre of mass. Further, let the space-time coordinates of the *proper* centre of mass in any K be denoted by $X_\mu = (\boldsymbol{X}, X_4)$, so that $\boldsymbol{X}(K^0) = \boldsymbol{X}^0$ in K^0. From the transformation properties of $\boldsymbol{u}$ it is apparent that all possible centres of mass have the same velocity $d\boldsymbol{X}/dt$ in any frame.

Let $m_{\mu\nu}$ represent the four-angular momentum components relative to the proper centre

$$m_{\mu\nu} = \int [(x_\mu - X_\mu) g_\nu - (x_\nu - X_\nu) g_\mu] dV = M_{\mu\nu} - (X_\mu G_\nu - X_\nu G_\mu). \tag{12.4}$$

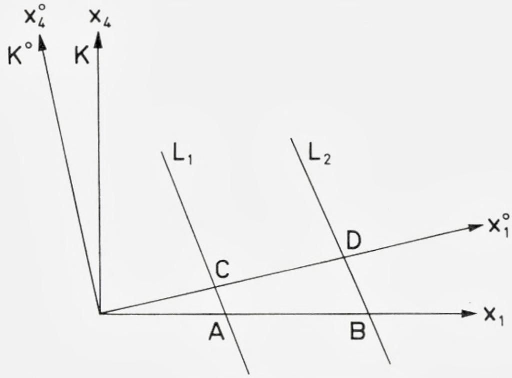

Fig. 2.

By differentiating the expression for $M_{\mu\nu}$ with respect to time along the moving wave elements where $d(g_\mu dV)/dt = 0$, we find that $dM_{\mu\nu}/dt = dx_\mu/dt \cdot G_\nu - dx_\nu/dt G_\mu$. Thus it follows that $dm_{\mu\nu}/dt = 0$ in any frame.

The difference $X_i(K) - X_i$ between simultaneous mass centres is in general related to m_{i4}:

$$m_{i4} = \frac{i}{c}\int (x_i - X_i) W dV = \frac{i}{c}[X_i(K) - X_i]\mathscr{H}. \tag{12.5}$$

Now the quantities $M_{\mu\nu}$ do not constitute a tensor. This follows from the fact that the quantities

$$\partial_\sigma(x_\mu S_{\nu\sigma} - x_\nu S_{\mu\sigma}) = S_{\nu\mu} - S_{\mu\nu} \tag{12.6}$$

in general do not vanish. (The detailed investigation of the tensor property of $M_{\mu\nu}$ goes similarly as the investigation of the four-vector property of G_μ, see § 63 of MØLLER's book(1).) Thus $m_{\mu\nu}$ cannot be obtained in K by a tensor transformation from K^0. This is a fundamental difference from the situation encountered for a closed system.

In order to find the actual coordinate difference we thus have to make an explicit calculation of the integrals in (12.5). In Fig. 2, L_1 and L_2 represent the cut with the x_1x_4-plane of a three-dimensional surface enclosing the field. Since $m_{\mu\nu}$ is a constant of motion, we choose to evaluate it in K at $t = 0$, i. e. along AB. Actually, we have to consider in detail only the first integral ($= M_{i4}$) in (12.5), for the second integral is equal to $-X_i G_4$ and G_4 is a component of a four-vector. We find readily

$$x_1(AB) = \gamma^{-1}x_1^0(AB), x_2(AB) = x_2^0(AB), x_3(AB) = x_3^0(AB), \tag{12.7}$$

and $W(AB)$ is related to the components $S^0_{\mu\nu}(AB)$ by a tensor transformation. Now seek to transform the integral over AB into an integral taken at constant time in K^0, and choose the domain CD for which $t^0 = 0$. This task can readily be accomplished for the internal, *plane* part of the radiation field. To this end we first observe that the world lines determined by $\boldsymbol{S}$ will each intersect AB and CD in two space-time points with coordinates $(x_i^0(AB), t^0(AB))$ and $(x_i^0(CD), 0)$ in K^0, such that

$$\left.\begin{aligned} x_1^0(CD) &= x_1^0(AB) - \frac{ck_1^0}{nk^0}t^0(AB) = x_1^0(AB)\left(1+\frac{\beta k_1^0}{nk^0}\right) \\ x_l^0(CD) &= x_l^0(AB) - \frac{ck_l^0}{nk^0}t^0(AB) = x_l^0(AB) + \frac{k_l^0}{k_0}\frac{\beta x_1^0(CD)}{n+\beta k_1^0/k^0} \quad (l = 2,3) \\ t^0(AB) &= -(\beta/c)x_1^0(AB). \end{aligned}\right\} \tag{12.8}$$

The volume integration in (12.5) shall be performed along the elements dV which follow the wave. Since the x_1-component of the wave velocity in K^0 is equal to $ck_1^0/(nk^0)$, the volume element dV is related to the corresponding element dV^0 taken at constant time in K^0 by

$$dV^0/dV = \gamma(1+\beta k_1^0/(nk^0)). \tag{12.9}$$

Further, we observe that $S^0_{\mu\nu}(AB) = S^0_{\mu\nu}(CD)$ at corresponding world points. For the internal plane wave part we have also

$$S^0_{ik} = W^0 k_i^0 k_k^0/k^{02}, \quad g_i^0 = nW^0 k_i^0/(ck^0). \tag{12.10}$$

When eqs. (12.7–10) are inserted into the expression for M_{l4}^{int}, i. e. the contribution to M_{l4} from the internal field, we obtain

$$M_{l4}^{\text{int}} = \frac{i\gamma}{c}\frac{1+n\beta k_1^0/k^0}{1+\beta k_1^0/(nk^0)}\left[\int_{\text{int}} x_l^0 W^0 dV^0 + \frac{v}{n^2}\int_{\text{int}}(x_l^0 g_1^0 - x_1^0 g_l^0)dV^0\right], \tag{12.11}$$

where the integrations are taken along CD, but only over the internal part of the wave.

Now it is apparent that, in addition to (12.11), we have to take into account also the effect from the thin boundary layer, which is responsible for the internal angular momentum of the field. This is in agreement with the fact that in the case of a *closed* system, the coordinate difference which we are seeking is connected with the total angular momentum in the inertial frame in which the total linear momentum vanishes. To investigate this boundary effect we introduce, purely formally, a tensor $S^S_{\mu\nu}$ which is defined by the following components in K^0:

$$S^{S0}_{i\nu} = S^0_{i\nu}/n^2, \ S^{S0}_{4\nu} = S^0_{4\nu}. \tag{12.12}$$

Thus the tensor $S^S_{\mu\nu}$ is symmetric and divergence-free, so that $M^S_{\mu\nu}$ is a tensor (cf. (12.6)). We immediately obtain

$$M^S_{l4} = \frac{i\gamma}{c}\int x^0_l W^0 dV^0 + i\beta\gamma\int (x^0_l g^{S0}_1 - x^0_1 g^{S0}_l) dV^0 \tag{12.13}$$

by a *tensor* transformation, where the integration domain includes the whole field. If we now calculate the internal part $M^{S,\,\mathrm{int}}_{l4}$ by transforming the integrand similarly as we did above, we find

$$M^{S,\,\mathrm{int}}_{l4} = \frac{i\gamma}{c}\int_{\mathrm{int}} x^0_l W^0 dV^0 + i\beta\gamma\int_{\mathrm{int}} (x^0_l g^{S0}_1 - x^0_1 g^{S0}_l) dV^0. \tag{12.14}$$

Here we have used eqs. (12.7–9) and the relations

$$S^{S0}_{ik} = W^0 k^0_i k^0_k/(n^2 k^{02}), \ g^{S0}_i = W^0 k^0_i/(nck^0), \tag{12.15}$$

which are valid for the internal part. By comparing (12.13) and (12.14) it is thus apparent that the total M^S_{l4} is obtained simply by extending the integration domain in (12.14) over the boundary region, such that g^{S0}_i and W^0 refer to the *total* momentum and energy densities in this region. (Actually, the additional term to the first integral in (12.14) is negligible.) Since g^{S0}_i is proportional to g^0_i, the same rule can be used to evaluate M_{l4} from (12.11), and we get

$$M_{l4} = \frac{i\gamma}{c}\,\frac{1 + n\beta k^0_1/k^0}{1 + \beta k^0_1/(nk^0)}\left(X^0_l \mathscr{H}^0 + \frac{v}{n^2} M^0_{l1}\right). \tag{12.16}$$

By means of (12.7), (12.8) and the transformation formula for $\mathscr{H}$ the last term in (12.5) is found as ($i = l$)

$$-\frac{i}{c} X_l \mathscr{H} = -\frac{i\gamma}{c}\,\frac{1 + n\beta k^0_1/k^0}{1 + \beta k^0_1/(nk^0)}\left[X^0_l \mathscr{H}^0 + \frac{v}{n^2}(X^0_l G^0_1 - X^0_1 G^0_l)\right], \tag{12.17}$$

where we have also used the relation $\boldsymbol{G}^0 = n\mathscr{H}^0 \boldsymbol{k}^0/(ck^0)$. The latter relation follows from the fact that the total linear momentum is obtained by integrating over the internal wave part. Then inserting (12.16), (12.17) into (12.5) and taking (12.4) into account, we find

$$m_{l4} = \frac{i\gamma\beta}{n^2} \frac{1 + n\beta k_1^0/k^0}{1 + \beta k_1^0/(nk^0)} m_{l1}^0 . \tag{12.18}$$

This is a boundary effect. Note that it is not necessary that the integral $\int x_l^0 W^0 dV^0$ over the boundary be taken as small in order to obtain (12.18), but when this boundary term is negligible, the expressions (12.17) and (12.11) are equal to each other, apart from a sign.

It can be verified that $X_l^0 G_1^0 - X_1^0 G_l^0$ is equal to L_{l1}^0 given by (11.7), and so m_{l1}^0 in (12.18) may be replaced by $\sum_{l1}^0$ given by (11.8).

Hitherto we have considered the cases $i = l = 2,3$. For $i = 1$ we obtain readily by the same method

$$M_{14} = \frac{i}{c} \frac{1 + n\beta k_1^0/k^0}{1 + \beta k_1^0/(nk^0)} X_1^0 \mathscr{H}^0, \tag{12.19a}$$

$$m_{14} = M_{14} - \frac{i}{c} X_1 \mathscr{H} = 0. \tag{12.19b}$$

By means of (12.5), (12.18) and (12.19) we can thus write the coordinate difference as

$$\boldsymbol{a}(K) = \boldsymbol{X}(K) - \boldsymbol{X} = \frac{\boldsymbol{v} \times \sum^0}{n^2(1 + \beta \cdot \boldsymbol{k}^0/(nk^0))\mathscr{H}^0}, \tag{12.20}$$

where $\sum^0$ is a vector with the components $\sum_i^0 = \sum_{jk}^0$ (i, j, k cyclic). The form (12.20) is obviously independent of the choice of the velocity vector $\boldsymbol{v}$ as lying along the x_1-axis. Since $\boldsymbol{a}(K)$ is perpendicular to $\boldsymbol{v}$, it will be left unchanged after a transformation from K to K^0.

Now we can calculate $\sum^0$ from (11.8) and find readily that $\sum^0/\mathscr{H}^0 = n\boldsymbol{k}^0/(ck^{02})$. By inserting this relation into (12.20) we get

$$\boldsymbol{a}(K) = \frac{1}{k^0} \frac{\beta \times \boldsymbol{k}^0}{nk^0 + \beta \cdot \boldsymbol{k}^0}. \tag{12.21}$$

Let us consider in K^0 the positions of the various centres of mass obtained by varying β and k^0 in (12.21). All centres lie in a plane perpendicular to $\boldsymbol{k}^0$, and if β, k^0 and $\beta \cdot \boldsymbol{k}^0$ are kept constant the end point of the vector $\boldsymbol{a}(K)$ will draw a circle with centre at the proper centre of mass.

The greatest radius of the circle is obtained when $\boldsymbol{k}^0 \cdot \beta = -k^0\beta^2/n$ and is equal to

$$a_{\max} = (\beta/nk^0)(1-\beta^2/n^2)^{-\frac{1}{2}}. \tag{12.22}$$

An arbitrary angle between β and $\boldsymbol{k}^0$ will in general lead to a centre of mass lying on the disk described by (12.22).

Permitting β to vary, we see that the greatest value of $a_{\max}$ occurs when $\beta = 1$. Further, $a_{\max} \to \infty$ when $k^0 \to 0$.

Instead of relating all centres of mass to the centre of mass in K^0, as seems to be most natural and as we have done in the present section, one may also relate these centres to the centre of mass in K^*, the frame in which the wave is at rest and the medium moving.

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